

USE OF COMPOUND MATRICES IN TOPOLOGICAL ANALYSIS

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## TABLE OF CONTENTS

I.	INTRODUCTION. . . . .	1
II.	BASIC DEFINITIONS AND THEOREMS. . . . .	4
	1. Graph Theory. . . . .	4
	2. Compound Matrices . . . . .	10
III.	COMPOUND MATRICES APPLIED TO THE TREE GENERATING PROBLEM . . . . .	17
	1. Finding Trees By The Compound Method. . . . .	17
	2. Upper Limit On Number of Minors . . . . .	22
	3. Number Trees of Distance k From the Starting Tree of a Graph . . . . .	24
IV.	SOME SIMPLIFICATIONS OF CALCULATIONS. . . . .	31
	1. Inspecting the Cotree for Circuits. . . . .	31
	2. Two Starting Trees. . . . .	32
	3. Pivotal Condensation. . . . .	36
	4. The Central Tree. . . . .	46
	5. Geometrical Interpretation Using Tree Graph . . . . .	57
V.	FINDING COMPLETE TREES OF ACTIVE NETWORKS . . . . .	63
VI.	CONCLUSIONS . . . . .	73
VII.	REFERENCES. . . . .	75
VIII.	ACKNOWLEDGMENTS . . . . .	77

## I. INTRODUCTION

In many applications, one would like to compute a network function in symbolic form in order to study the effect of parameter variations. Topological formulas for network functions are very convenient for this purpose [1-15]. They are more efficient than the classical methods involving calculating the inverse of a nodal or mesh matrix because the network functions are generated without handling terms which ultimately cancel. Also, the formulas are easily adapted to machine calculations. These formulas require that one list all trees of a linear graph, and in the case of networks containing active elements or transformers, compute the algebraic signs associated with certain tree pairs. Many methods of finding trees and their signs have been developed [7, 8, 10, 12, 13, 15], however there remains an urgent need for more efficient methods.

In this report, the problem of finding trees and their signs is approached by means of compound matrices. The principal advantages of the compound method are a compact notation and high degree of organization. In active network problems, the signs of the appropriate tree-pairs are generated as part of the tree-finding procedure, a separate calculation being unnecessary. The compound method organizes the tree-finding problem in such a manner that one can readily find specified subsets of the set of trees of a graph. For example, it is quite easy to find the set

of all trees which contain a given set of edges and do not contain a given set of edges.

It is well-known that in the f-cut-set matrix,  $Q_f = [U \ Q]$ , the nonzero entries of  $Q$  are in one-to-one correspondence with trees of distance one from the tree  $t$  used to determine  $Q_f$  [13]. Tree  $t$  is called the starting tree [13] since all other trees are found using this tree. The new method shows that the nonzero entries of  $Q^{(k)}$ , the  $k$ th compound of  $Q$ , are in one-to-one correspondence with trees of distance  $k$  from the starting tree. Thus, all the trees of a linear graph can be found by inspection of  $Q$  and a certain number of its compounds. The information necessary for finding the relative signs of the trees is also included in the compounds.

The basic compound method is not a particularly efficient tree-generating procedure; however, some methods for reducing the number of necessary calculations and their complexity have been developed. The principle of pivotal condensation can be used to calculate the  $m$ th compound in terms of minors of order 2 of the  $m-1$  compound and pivot elements from the  $m-2$  compound. This makes it possible to locate all trees of a graph by using only determinants of order two. The number of calculations can be reduced by properly choosing the starting tree and by an inspection of the cotree of the starting tree. Perhaps most important, methods of reducing the number of calculations in the compound method are often applicable to the more efficient tree-finding procedures. That is, the insight and overview

provided by the less-efficient compound method leads to simplifications of the more efficient methods which are not obvious when these methods are studied alone.

## II. BASIC DEFINITIONS AND THEOREMS

This chapter consists of a list of definitions and theorems. The items in the graph theory section are all well-known in the circuit theory literature. This list is limited to those items which are required for a lucid development of the new concepts contained in subsequent chapters. Any terms and concepts not specifically defined here may be found in [9]. The items in the section on compound matrices are relatively new to the circuit theory literature. All of these may be found in the mathematics literature, with the exception of Theorem 2-13, which is proved here for the first time and illustrated by means of an example. The application of this theorem to network problems is deferred until Chapter IV. The proofs of all other theorems stated in this chapter may be found in the references cited.

### 1. Graph Theory

Definition 2-1 [9] A line segment together with distinct endpoints is an edge.

Definition 2-2 [9] A vertex is an endpoint of an edge.

Definition 2-3 [9] A vertex and an edge are incident with each other if the vertex is an endpoint of the edge.

Definition 2-4 [9] The degree of a vertex is the number of edges incident at the vertex.

Definition 2-5 [9] A path is a sequence of edges of a graph having exactly two (terminal) vertices of degree one and all other (internal) vertices of degree two.

Definition 2-6 [9] A graph is connected if for every vertex pair  $(v_i, v_j)$  in  $G$ , there exists a path in  $G$  having  $v_i$  and  $v_j$  as terminal vertices.

Definition 2-7 [9] The length of a path is the number of edges contained in the path.

Definition 2-8 [9] The distance,  $d(v_i, v_j)$ , between vertices  $v_i$  and  $v_j$  of a connected graph is the length of the shortest path in the graph having  $v_i$  and  $v_j$  as endpoints.

Definition 2-9 [21] For a fixed vertex  $v$  of a connected graph  $G$ , the integer

$$R(v) = \max_{v_i \in G} d(v, v_i)$$

where  $\max_{v_i \in G}$  denotes maximizing the distance over all vertices of  $G$ , measures the distance from  $v$  to the vertex most remote from  $v$ . It is intuitively clear that a vertex is relatively central if  $R(v)$  is relatively small. Then it is natural to call

$$R_0 = \min_{v \in G} R(v)$$

the radius of  $G$ , and to refer to any vertex  $v_0$  as a center of  $G$  if

$$R(v_0) = R_0$$

It is well-known that the center of a graph is not unique.

Definition 2-10 [21] The diameter of a connected graph is the maximum distance between pairs of vertices. That is

$$\text{Diameter} = \max_{v_i, v_j \in G} d(v_i, v_j)$$

Definition 2-11 [9] An oriented edge is an edge with an orientation assigned by ordering its vertices.

Definition 2-12 [9] A graph in which every edge has been assigned an orientation is directed graph.

Definition 2-13 [21] A graph is said to be simple if it contains no parallel edges.

Definition 2-14 [21] A simple graph is said to be complete if every distinct pair of vertices are joined by an edge. Furthermore, the total number of edges  $e$  of a complete graph having  $v$  vertices is given by

$$e = \frac{v(v-1)}{2}$$

Definition 2-15 [9] A tree is a connected subgraph of a connected graph which contains all the vertices of the graph but does not contain any circuits.

Definition 2-16 [9] The nullity  $\mu$  of a graph with  $v$  vertices,  $e$  edges and  $p$  maximal connected subgraphs is defined by  $\mu = e - v + p$ . For a connected graph,  $\mu = e - v + 1$ .

Definition 2-17 [9] The rank  $r$  of a graph with  $v$  vertices and  $p$  maximal connected subgraphs is given by  $r = v - p$ . For a connected graph,  $r = v - 1$ .

Theorem 2-1 [9]

A subgraph  $G_s$  of a connected graph  $G$  can be made part of tree if and only if  $G_s$  contains no circuits.



Definition 2-18 [9] Two trees of a connected graph  $G$  having  $v$  vertices are at distance  $k$  when there are exactly  $v - 1 - k$  edges of  $G$  common to the two trees of  $G$ .

Definition 2-19 [9] A cut-set is a set of edges of a connected graph  $G$  such that the removal of these edges from  $G$  reduces the rank of  $G$  by one, provided that no proper subset of this reduces the rank of  $G$  by one when it is removed from  $G$ .

In discussing cut-sets, it is conceptually convenient to regard an edge as open, that is, as not including its endpoints. Thus any isolated vertices of a graph are included in the number of maximal connected subgraphs. This viewpoint applies to all discussions which follow.

Definition 2-20 [9] The fundament system of cut-sets with respect to a tree  $t$  is the set of  $v-1$  cut-sets, one for each branch, in which each cut-set includes exactly one branch of  $t$ .

Definition 2-21 [9] A graph is planar if it can be mapped onto a plane such that no two edges have a point in common that is not a vertex. Otherwise, the graph is nonplanar.

Definition 2-22 [9] Graph  $G_2$  is a dual of  $G_1$ , if there is a one-to-one correspondence between the edges of the two graphs  $G_1$  and  $G_2$  such that if  $H_1$  is any subgraph of  $G_1$  and  $H_2$  is the complement of the corresponding subgraph of  $G_2$ , then

$$r_2 = R_2 - n_1$$

where  $r_2$  and  $R_2$  are rank of  $H_2$  and  $G_2$ , respectively and  $n_1$  is the nullity of  $H_1$ .

## Theorem 2-2 [9]

Let  $G_2$  be a dual of  $G_1$ , then

$$R_1 = N_2 \quad \text{and} \quad R_2 = N_1$$

where  $R_1$  and  $R_2$  are ranks of  $G_1$  and  $G_2$ , respectively, and  $N_1$  and  $N_2$  are the nullities of  $G_1$  and  $G_2$ , respectively.

## Theorem 2-3 [9]

A graph has a dual if and only if it is planar.

Let  $G$  be a connected graph of  $v$  vertices and  $e$  edges.

Definition 2-23 [9] The incidence matrix,  $A_a = [a_{ij}]$  of  $G$  is a matrix of  $v$  rows and  $e$  columns such that for an undirected graph:

$a_{ij} = 1$ , if edge  $j$  is incident at vertex  $i$ ,

$a_{ij} = 0$ , if edge  $j$  is not incident at vertex  $i$ .

For a directed graph:

$a_{ij} = 1$ , if edge  $j$  is incident at vertex  $i$  and is oriented away from vertex  $i$ ,

$a_{ij} = -1$ , if edge  $j$  is incident at vertex  $i$  and is oriented toward vertex  $i$ ,

$a_{ij} = 0$ , if edge  $j$  is not incident at vertex  $i$ .

## Theorem 2-4 [9]

The rank of the vertex matrix  $A_a$  of  $G$  is  $v - 1$ .

## Corollary 2-4 [9]

If any row of the matrix  $A_a$  of  $G$  is deleted, the resulting matrix  $A$  has a rank of  $v-1$ . The matrix  $A$  is also referred to as the incidence matrix of  $G$ .

## Theorem 2-5 [9]

The determinant of every nonsingular submatrix of  $A$  is 1 for an undirected graph,  $\pm 1$  for a directed graph.

## Theorem 2-6 [9]

A square submatrix of  $A$  of order  $v-1$  is nonsingular if and only if the elements corresponding to the columns of this submatrix constitute a tree of  $G$ .

## Corollary 2-6 [9]

The trees of  $G$  are in one-to-one correspondence with the nonsingular submatrices of order  $v-1$  of  $A$ .

Definition 2-24 [9] The cut-set matrix  $Q_a = [q_{ij}]$  has one row for each possible cut-set and one column for each edge, and is such that for an undirected graph:

$$q_{ij} = 1, \text{ if edge } j \text{ is in cut-set } i,$$

$$q_{ij} = 0, \text{ if edge } j \text{ is not in cut-set } i.$$

For a directed graph:

$$q_{ij} = 1, \text{ if edge } j \text{ is in cut-set } i \text{ and the orientations agree,}$$

$$q_{ij} = -1, \text{ if edge } j \text{ is in cut-set } i \text{ and the orientations are opposite,}$$

$$q_{ij} = 0, \text{ if edge } j \text{ is not in cut-set } i.$$

## Theorem 2-7 [9]

For every submatrix  $\underline{Q}$  of  $v-1$  rows and  $e$  columns and of rank  $v-1$  selected from a cut-set matrix  $Q_a$ , the nonsingular submatrices of  $\underline{Q}$  of order  $v-1$  are in one-to-one correspondence with the trees of  $G$ .

Definition 2-25 [9] Let  $t$  be a tree of a connected graph  $G$ . The f-cut-set matrix  $Q_f$  with respect to  $t$  is a cut-set matrix  $Q$  such that each row of  $Q_f$  corresponds to the unique cut-set defined by a branch of  $t$ . By ordering the columns according to branches first and then chords,  $Q_f$  can always be written in the form

$$Q_f = [U \quad Q]$$

where  $U$  is a unity matrix of order  $v-1$ .

Theorem 2-8 [9]

If the columns of the matrix  $A$  and  $Q_f$  are arranged in the order of branches and chords for the tree  $t$  for which the fundamental systems are formed, and then partitioned as

$$A = [A_t \quad A_c] \quad \text{and} \quad Q_f = [U \quad Q]$$

then

$$Q_f = A_t^{-1} A \quad \text{and} \quad Q = A_t^{-1} A_c.$$

Theorem 2-9 [13]

If  $Q_f = [U \quad Q]$  is an f-cut-set matrix of a connected graph  $G$  with respect to a tree  $t$ , then the nonzero entries of  $Q$  are in one-to-one correspondence with the trees of the graph of distance one from  $t$ .

## 2. Compound Matrices

Definition 2-26 [18] Let  $M$  be a matrix of order  $m \times n$ , and let  $l(m, n)$  denote the smaller of the numbers  $m$  and  $n$  for  $m \neq n$ ; with  $l(m, n) = n$  for  $m = n$ . Let  $\binom{p}{s}$  denote the number of distinct

combinations of  $p$  objects taken  $s$  at a time. The  $k$ th compound of  $M$ ,  $M^{(k)}$ ,  $1 \leq k \leq l(m, n)$ , is the matrix of order  $\binom{m}{k} \times \binom{n}{k}$  having all minors of  $M$  of order  $k$  for elements. All minors formed from the same group of  $k$  rows (columns) of  $M$  are placed in the same row (column) of  $M^{(k)}$  and arranged in lexical order, that is, the same order in which words are arranged in a dictionary or lexicon.

By definition,  $M^{(1)} = M$ . It is easy to prove that for a matrix  $M$  and scalar  $a$ ,

$$(aM)^{(k)} = a^k M^{(k)} \quad (2-1)$$

Definition 2-27 [18] The  $k$ th adjugate compound,  $\text{adj}^{(k)} M$ , of a square matrix  $M$  of order  $n$  is formed by replacing every element of  $M^{(k)}$  by the algebraic complement associated with that minor in Laplace's expansion of  $|M|$  and by transposing the resulting matrix. Clearly,  $\text{adj}^{(k)} M$  is of order  $\binom{n}{k} \times \binom{n}{k}$ , with  $\text{adj}^{(1)} M$  being the ordinary adjoint matrix.

Theorem 2-10 [18]

If  $M$  and  $N$  are matrices of order  $m \times n$  and  $n \times p$ , respectively, then

$$(M N)^{(k)} = M^{(k)} N^{(k)} \quad , \quad 1 \leq k \leq l(m, n, p), \quad (2-2)$$

where  $l(m, n, p)$  is the smallest of the numbers  $m$ ,  $n$ , and  $p$ .

Theorem 2-11 [18]

If  $M$  is square of order  $n$ , then

$$(\text{adj } M)^{(k)} = (\det M)^{k-1} \text{adj}^{(k)} M, \quad 1 \leq k \leq n. \quad (2-3)$$

This holds even when  $M$  singular provided then when  $k = 1$  the interpretation  $(\det M)^0 = 1$  is used.

Theorem 2-12 [19]

If  $M$  is square of order  $n$ , then

$$[M^{(k)}]_{n_k - q + 1, n_k - p + 1} = (-1)^s [\text{adj}^{(n-k)} M]_{p, q} \quad (2-4)$$

where the symbolism  $[N]_{i, j}$  denotes the element in row  $i$  and column  $j$  of matrix  $N$ , and where  $n_k$  is a more concise expression for  $\binom{n}{k}$ . In (2-4), the left side is some particular minor of  $M$  of order  $k$ ;  $s$  denotes the sum of the row and column indices of this minor in  $M$ . In Bryant's paper this sign is incorrectly associated with the row and column indices of the compound element.

The following theorem relates the higher compounds of a nonsingular matrix to the lower compounds of the inverse of the matrix. Thus the twelfth compound of a matrix  $M$  of order thirteen, for example, can be found from the first compound of  $M^{-1}$ , obviating the need for calculating minors of  $M$  of order twelve.

Theorem 2-13

Let two nonsingular square matrices  $M_1$  and  $M_2$  of order  $n$  satisfy  $M_1 M_2 = U$ , where  $U$  denotes a unity matrix. Then

$$[M_1^{(n-k)}]_{p, q} = (-1)^s (\det M_2)^{-1} [M_2^{(k)}]_{n_k - q + 1, n_k - p + 1} \quad (2-5)$$

for  $1 \leq k < n-1$ .

Proof: Since  $M_1 M_2 = U$ ,

$$M_1 = M_2^{-1} = (\det M_2)^{-1} [\text{adj } M_2]. \quad (2-6)$$

Taking the  $(n-k)$ th compound of both sides of (2-6) gives

$$M_1^{(n-k)} = (\det M_2)^{-(n-k)} [\text{adj } M_2]^{(n-k)} . \quad (2-7)$$

By Theorem 2-11,

$$\begin{aligned} M_1^{(n-k)} &= (\det M_2)^{-(n-k)} (\det M_2)^{n-k-1} [\text{adj }^{(n-k)} M_2] \\ &= (\det M_2)^{-1} [\text{adj }^{(n-k)} M_2] . \end{aligned} \quad (2-8)$$

Substituting (2-4) into (2-8) gives

$$\begin{aligned} [M_1^{(n-k)}]_{p,q} &= (\det M_2)^{-1} [\text{adj }^{(n-k)} M_2]_{p,q} \\ &= (\det M_2)^{-1} (-1)^s [M_2^{(k)}]_{n_k - q + 1, n_k - p + 1} \end{aligned} \quad (2-9)$$

which completes the proof.

In words,  $M_1^{(n-k)}$  can be obtained by reversing the order of both rows and columns of  $k$ th compound of the transposed matrix  $M_2$ , and by then multiplying each element with an appropriate signed scalar multiplier.

As an example, consider the following two matrices  $M_1$  and  $M_2$ , which satisfy the relation  $M_1 M_2 = U$ , with  $\det M_2 = \frac{1}{15}$ .

$$M_1^{(1)} = \begin{matrix} & \begin{matrix} f & g & h & i & j \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 3 & 3 & -1 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 2 & -1 & 2 & 2 \end{bmatrix} \end{matrix} \quad (2-10)$$

$$M_2^{(1)} = \left(\frac{1}{15}\right) \begin{matrix} & \begin{matrix} a' & b' & c' & d' & e' \end{matrix} \\ \begin{matrix} f' \\ g' \\ h' \\ i' \\ j' \end{matrix} & \begin{bmatrix} 30 & -20 & -15 & 25 & -5 \\ 30 & -11 & -18 & 7 & -8 \\ -30 & 12 & 21 & -9 & 6 \\ -15 & 12 & 6 & -9 & 6 \\ 15 & -7 & -6 & -1 & -1 \end{bmatrix} \end{matrix} \quad (2-11)$$

According to Theorem 2-13, the third compound of  $M$  can be found from the second compound of  $M_2$ , by the following procedure.

First find the transpose

$$M_2^T = \left(\frac{1}{15}\right) \begin{matrix} & \begin{matrix} f' & g' & h' & i' & j' \end{matrix} \\ \begin{matrix} a' \\ b' \\ c' \\ d' \\ e' \end{matrix} & \begin{bmatrix} 30 & 30 & -30 & -15 & 15 \\ -20 & -11 & 12 & 12 & -7 \\ -15 & -18 & 21 & 6 & -6 \\ 25 & 7 & -9 & -9 & -1 \\ -5 & -8 & 6 & 6 & -1 \end{bmatrix} \end{matrix} \quad (2-12)$$

The second compound is then

$$(M_2^T)^{(2)} = \left(\frac{1}{(15)^2}\right) \begin{matrix} & \begin{matrix} f'g' & f'h' & f'i' & f'j' & g'h' & g'i' & g'j' & h'i' & h'j' & i'j' \end{matrix} \\ \begin{matrix} a'b' \\ a'c' \\ a'd' \\ a'e' \\ b'c' \\ b'd' \\ b'e' \\ c'd' \\ c'e' \\ d'e' \end{matrix} & \begin{bmatrix} 270 & -240 & 60 & 90 & 30 & 195 & -45 & -180 & 30 & -75 \\ -90 & 180 & -45 & 45 & 90 & -90 & 90 & 135 & -135 & 0 \\ -540 & 480 & 105 & -405 & -60 & -165 & -135 & 135 & 165 & 150 \\ -90 & 30 & 105 & 45 & -60 & 60 & 90 & -90 & -60 & -75 \\ 195 & -240 & 60 & 15 & -15 & 150 & -60 & -180 & 75 & -30 \\ 135 & -120 & -120 & 195 & 15 & 15 & 60 & 0 & -75 & -75 \\ 105 & -60 & -60 & -15 & 30 & 30 & -45 & 0 & 30 & 30 \\ 345 & -390 & -15 & 165 & 15 & 150 & 60 & -135 & -75 & -60 \\ 30 & 15 & -60 & -15 & 60 & -60 & -30 & 90 & 15 & 30 \\ -165 & 105 & 105 & -30 & -30 & -30 & -15 & 0 & 15 & 15 \end{bmatrix} \end{matrix} \quad (2-13)$$



Reversing the order of the rows and columns of  $(M_2^T)^{(2)}$  gives,

$$\left(\frac{1}{(15)^2}\right) \begin{matrix} & i'j' & h'j' & h'i' & g'j' & g'i' & g'h' & f'j' & f'i' & f'h' & f'g' \\ \begin{matrix} d'e' \\ c'e' \\ c'd' \\ b'e' \\ b'd' \\ b'c' \\ a'e' \\ a'd' \\ a'c' \\ a'b' \end{matrix} & \begin{bmatrix} 15 & 15 & 0 & -15 & -30 & -30 & -30 & 105 & 105 & -165 \\ 30 & 15 & 90 & -30 & -60 & 60 & -15 & -60 & 15 & 30 \\ -60 & -75 & -135 & 60 & 150 & 15 & 165 & -15 & -390 & 345 \\ 30 & 30 & 0 & -45 & 30 & 30 & -15 & -60 & -60 & 105 \\ -75 & -75 & 0 & 60 & 15 & 15 & 195 & -120 & -120 & 135 \\ -30 & 75 & -180 & -60 & 150 & -15 & 15 & 60 & -120 & 195 \\ -75 & -60 & -90 & 90 & 60 & -60 & 45 & 105 & -240 & -90 \\ 150 & 165 & 135 & -135 & -165 & -60 & -405 & 105 & 30 & -540 \\ 0 & -135 & 135 & 90 & -90 & 90 & 45 & -45 & 180 & -90 \\ -75 & 30 & -180 & -45 & 195 & 30 & 90 & 60 & -240 & 270 \end{bmatrix} \end{matrix}.$$

(2-14)

Multiplying this matrix by  $(\det M_2)^{-1} = 15$ , and appropriately adjusting signs gives

$$\begin{matrix} & i'j' & h'j' & h'i' & g'j' & g'i' & g'h' & f'j' & f'i' & f'h' & f'g' \\ \begin{matrix} d'e' \\ c'e' \\ c'd' \\ b'e' \\ b'd' \\ b'c' \\ a'e' \\ a'd' \\ a'c' \\ a'b' \end{matrix} & \begin{bmatrix} 1 & -1 & 0 & -1 & 2 & -2 & 2 & 7 & -7 & -11 \\ -2 & 1 & -6 & 2 & -4 & -4 & -1 & 4 & 1 & -2 \\ -4 & 5 & -9 & 4 & -10 & 1 & -11 & -1 & 26 & 23 \\ 2 & -2 & 0 & -3 & -2 & 2 & 1 & -4 & 4 & 7 \\ 5 & -5 & 0 & -4 & 1 & -1 & 13 & 8 & -8 & -9 \\ -2 & -5 & -12 & -4 & 10 & -1 & -1 & 4 & 16 & 13 \\ 5 & -4 & 6 & -6 & 4 & 4 & 3 & -7 & 2 & 6 \\ 10 & -11 & 9 & -9 & 11 & -4 & 27 & 7 & -32 & 36 \\ 0 & -9 & -9 & -6 & -6 & -6 & 3 & 3 & 12 & 6 \\ -5 & -2 & -12 & -3 & 13 & 2 & -6 & 4 & 16 & 18 \end{bmatrix} \end{matrix} . \quad (2-15)$$

Using the definition of the third compound,  $M_1^{(3)}$  is found by direct calculation to be

	fgh	fgi	fgj	fhi	fhj	fij	ghi	ghj	gij	hij	
abc	1	-1	0	-1	2	-2	2	7	-7	-11	(2-16)
abd	-2	1	-6	2	-4	-4	-1	4	1	-2	
abe	-4	5	-9	4	-10	1	-11	-1	26	23	
acd	2	-2	0	-3	-2	2	1	-4	4	7	
ace	5	-5	0	-4	1	-1	13	8	-8	-9	
ade	-2	-5	-12	-4	10	-1	-1	4	16	13	
bcd	5	-4	6	-6	4	4	3	-7	2	6	
bce	10	-11	9	-9	11	-4	27	7	-32	36	
bce	0	-9	-9	-6	-6	-6	3	3	12	6	
cde	-5	-2	-12	-3	13	2	-6	4	16	18	

which is exactly the matrix (2-15).

At this point, the row and column labeling of the matrices serves only to relate the elements of the compounds to the appropriate minors and to clarify the row and column operations. After the applications of Theorem 2-13 are developed in Chapter IV, it will be helpful to review this example and make the following additional observation. The row (column) label of any entry in (2-16) may be found by removing the primes from the corresponding label in (2-15) and by then complementing this result with respect to the set of row (column) labels of (2-10).

### III. COMPOUND MATRICES APPLIED TO THE TREE GENERATING PROBLEM

In this chapter, compound matrix theory is applied to the problem of finding the trees of a connected graph. It is first proved that there exists a one-to-one correspondence between the nonzero entries in the  $k$ th compound of a certain matrix  $Q$  and the trees of distance  $k$  from a known starting tree of the graph. It is also proved that inspecting all of the defined compounds of  $Q$  is sufficient for finding all trees of the graph. Since finding trees by this method involves calculating minors of  $Q$ , an upper limit on the number of calculations necessary to find all of the trees is derived. Finally, expressions are given for the number of elements contained in certain subsets of the set of all trees of a graph.

#### 1. Finding Trees By The Compound Method

Let  $G$  be a connected linear graph of rank  $r$  and nullity  $\mu$ , and let  $t$  be a tree of  $G$ . The  $r$  rowed incidence matrix of  $G$  may be written in the partitioned form,

$$A = [A_t \ A_c] \quad (3-1)$$

where the columns of  $A_t$  correspond to branches of  $t$ .

The  $f$ -cut-set matrix with respect to  $t$  is then,

$$Q_f = [U \ Q] = [U \ A_t^{-1} \ A_c] \quad (3-2)$$

where  $Q$  is of order  $r \times \mu$ .

#### Theorem 3-1

All trees of  $G$  can be found without duplication by inspection of the nonzero elements in compounds one through  $l(r, \mu)$  of  $Q$ .

Proof:

Using (3-2) and Theorem 2-10,

$$\begin{aligned} Q^{(k)} &= [A_t^{-1} A_c]^{(k)} \\ &= (\det A_t)^{-k} (\text{adj } A_t)^{(k)} A_c^{(k)}. \end{aligned} \quad (3-3)$$

Using Theorems 2-5 and 2-11, (3-3) becomes

$$Q^{(k)} = (\det A_t) (\text{adj } A_t)^{(k)} A_c^{(k)}. \quad (3-4)$$

Now examine element  $[Q^{(k)}]_{ij}$  of  $Q^{(k)}$  in (3-4). By Definition 2-27, the elements of the  $i$ th row of  $(\text{adj } A_t)^{(k)}$  are the algebraic complements of the minors of order  $k$  formed from the  $i$ th set of  $k$  columns of  $A_t$ , the minors being arranged in lexical order. By Definition 2-26, the elements in the  $j$ th column of  $A_c^{(k)}$  are the minors of order  $k$  formed from the  $j$ th set of  $k$  columns of  $A_c$ , the minors again being arranged in lexical order. Thus,  $[Q^{(k)}]_{ij}$  is  $(\det A_t)$  times the determinant of that matrix,  $A_x$ , formed by replacing a certain  $k$  columns of  $A_t$  by a certain  $k$  columns of  $A_c$ . By Theorem 2-6,  $A_x$  is nonsingular if and only if its columns correspond to a tree of  $G$ . Furthermore if the columns of  $A_x$  correspond to a tree of  $G$ ,  $t_x$  is of distance  $k$  from  $t$ . Thus, there is a one-to-one correspondence between the nonzero elements in  $Q^{(k)}$  and the trees of  $G$  of distance  $k$  from  $t$ . Because of this correspondence, all trees of distance  $k$  can be found without duplication. To see that all trees of  $G$  may be found from compounds one through  $l(r, \mu)$  of  $G$ , first consider the case  $r \leq \mu$ , where  $l(r, \mu) = r$ . A tree of greatest possible distance from  $t$  is found by replacing all  $r$  branches of  $t$  by chords.

Thus, the greatest possible distance is  $r$ . Now consider the case  $r > \mu$ , where  $l(r, \mu) = \mu$ . Since  $t$  has only  $\mu$  chords, every other tree of  $G$  must contain  $r - \mu$  branches of  $t$ . Thus, the greatest possible distance of a tree from  $t$  is  $\mu$ . This completes the proof.

The preceding derivation holds for matrices defined over either the field of real numbers or the field modulo 2. In passive network problems it is convenient to deal with undirected graphs using modulo 2 algebra. In active network problems, the algebra of real numbers must be used. The following examples illustrate how trees are found for both undirected and directed graphs.

Example 3-1. Consider the graph of Figure 3-1. Let the starting tree be  $t = abe$ . Then

$$Q^{(1)} = \begin{matrix} & c & d & f \\ \begin{matrix} a \\ b \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix} \quad (3-5)$$

$$Q^{(2)} = \begin{matrix} & cd & cf & df \\ \begin{matrix} ab \\ ae \\ be \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix} \quad (3-6)$$

$$Q^{(3)} = \begin{matrix} & cdf \\ \begin{matrix} abe \end{matrix} & [ 1 ] \end{matrix} \quad (3-7)$$

The algebra of the field modulo 2 was used to find the entries of compounds  $Q^{(2)}$  and  $Q^{(3)}$ . A nonzero entry such as the

(b,c) entry of  $Q^{(1)}$  corresponds to the tree ace of distance one from t which is obtained by replacing branch b of t by edge c of the cotree. Using this procedure, the other six trees of distance one from t are found to be:

abc, dbc, ade, fbe, afe, and abf.

Similarly, from  $Q^{(2)}$  the seven trees of distance two from t are found to be:

cde, cbd, acd, cfe, cbf, dbf, and adf.

From  $Q^{(3)}$  cdf is found to be a tree of distance three from t. Therefore the complete set of trees of the graph are the sixteen trees listed above including the starting tree t.

Example 3-2. Consider the directed graph of Figure 3-2. Let adeg be the starting tree t. Then

$$Q^{(1)} = \begin{matrix} & \begin{matrix} b & c & f & i & j \end{matrix} \\ \begin{matrix} a \\ d \\ e \\ g \\ h \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \end{matrix} \quad (3-8)$$

$$Q^{(2)} = \begin{matrix} & bc & bf & bi & bj & cf & ci & cj & fi & fj & ij \\ \begin{matrix} ad \\ ae \\ ag \\ ah \\ de \\ dg \\ dh \\ eg \\ eh \\ gh \end{matrix} & \left[ \begin{array}{ccccccccc} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{array} \right] \end{matrix} \quad (3-9)$$

$$Q^{(3)} = \begin{matrix} & bdf & bci & bcj & bfi & bfj & bij & cfi & cfj & cij & fij \\ \begin{matrix} ade \\ adg \\ adh \\ aeg \\ aeh \\ agh \\ deg \\ deh \\ dgh \\ egh \end{matrix} & \left[ \begin{array}{cccccccccc} 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array} \right] \end{matrix} \quad (3-10)$$

In  $Q^{(2)}$ , the nonzero element in position (ae, bi) denotes the tree formed by replacing branches a and e of t by chords b and i; that is, tree bdigh which is of distance two from t. In this manner, all trees of G may be found.

## 2. Upper Limit On Number of Minors

It was shown in the last section that all of the trees of a graph can be found from a starting tree  $t$  by calculating compounds one through  $l(r, \mu)$  of  $Q$ . Assuming  $Q$  is given, an upper limit on the number of determinants which must be calculated to find all of the trees is found by calculating the number,  $n$ , of elements contained in compounds two through  $l(r, \mu)$  of  $Q$ . By the definition of the  $k$ th compound,

$$n = \sum_{k=2}^{l(r, \mu)} \binom{r}{k} \binom{\mu}{k} . \quad (3-11)$$

By the Binomial Theorem, the sum of the products may be found as follows. Assume that  $r \leq \mu$ . Then, for a pair of parameters  $a$  and  $b$ ,

$$(a + b)^{r+\mu} = (a + b)^r (a + b)^\mu \quad (3-12)$$

which gives

$$\sum_{x=0}^{r+\mu} \binom{r+\mu}{x} a^x b^{r+\mu-x} = \sum_{k=0}^r \binom{r}{k} a^k b^{r-k} \sum_{y=0}^{\mu} \binom{\mu}{y} a^y b^{\mu-y} . \quad (3-13)$$

From both sides of (3-13), select the term involving  $a^z b^{r+\mu-z}$ . Note that this implies  $z = k+y$  on the right side of (3-13). Then

$$\binom{r+\mu}{z} a^z b^{r+\mu-z} = \sum_{k=0}^r \binom{r}{k} \binom{\mu}{z-k} a^z b^{r+\mu-z} . \quad (3-14)$$

Thus



$$\binom{r+\mu}{z} = \sum_{k=0}^r \binom{r}{k} \binom{\mu}{z-k} \quad (3-15)$$

For the case that  $r > \mu$ , the symbols for  $r$  and  $\mu$  are interchanged and the derivation is repeated. Using these results and letting  $z=r$  or  $z=\mu$  gives

$$\binom{r+\mu}{\mu} = \binom{r+\mu}{r} = \sum_{k=0}^1 \binom{r, \mu}{k} \binom{r}{k} \binom{\mu}{k} \quad (3-16)$$

Therefore, we have

$$n = \binom{r+\mu}{r} - \binom{r}{1} \binom{\mu}{1} - \binom{r}{0} \binom{\mu}{0} \quad (3-17)$$

The term  $\binom{r+\mu}{r}$  in (3-17) is the number of majors which may be formed from the incidence matrix  $A$  of the graph. This is diminished by  $\binom{r}{0} \binom{\mu}{0} = 1$  calculation which is needed to find the determinant of the incidence matrix of the starting tree and by  $\binom{r}{0} \binom{\mu}{0} = r\mu$  calculations which is the number of minors needed in order to find  $Q$  in the special case  $r=\mu$ . The point is that the total number of minors which are required in calculating  $Q$  and all of its compounds is not necessarily less than the number of minors of  $A$  which must be calculated in an exhaustive trial of all sets of  $r$  edges of  $G$ . It should be noted, however, that only when  $r \leq \mu$  is it necessary to calculate minors of order  $r$ ; and even then only  $\binom{\mu}{r}$  minors of this order are required. All other minors are of order lower than  $r$  in the compound method. Some methods for reducing the number of calculations are discussed in the following chapter. In particular, it will be shown that

although  $l(r,u)$  compounds are sufficient for finding all of the trees, a smaller number will often do if the starting tree is properly selected.

### 3. Number Trees of Distance $k$ From the Starting Tree of a Graph

In a new and more general set of active network topological formulas recently derived [24], efficient calculation of network functions requires that sets of trees containing certain specified groups of edges be located. If such tree sets are to be located by digital computer, an a-priori knowledge of the number of trees in a given set makes it possible to improve the efficiency of calculations by eliminating the possibility of searching for additional members of an already completed set. The following theorem, applicable only to directed graphs, is expected to be very useful in such applications.

Theorem 3-2.

The number of distinct trees which can be obtained by replacing the  $i$ th set of  $k$  branches of the starting tree by sets of  $k$  chords is given by

$$[(Q Q^T)^{(k)}]_{i,i} \quad (3-18)$$

The number of distinct trees which can be obtained by replacing sets of  $k$  branches of the starting tree by the  $i$ th set of  $k$  chords is given by

$$[(Q^T Q)^{(k)}]_{i,i} \quad (3-19)$$

Proof: From Theorem 2-10,

$$(Q Q^T)^{(k)} = Q^{(k)} (Q^{(k)})^T \quad (3-20)$$

As shown previously, every row of  $Q^{(k)}$  corresponds to a certain set of  $k$  branches of the starting tree. Row  $i$  corresponds to the  $i$ th set of  $k$  branches. By the derivation in previous section, the number of nonzero entries of row  $i$  of  $Q^{(k)}$  exactly equals the number of trees which can be formed by replacing the  $i$ th set of  $k$  branches by chords. By the proof of Theorem 2-5, every nonzero entry of this row is either  $+1$  or  $-1$ . Since the  $i$ th column of  $(Q^{(k)})^T$  contains the same entries as the  $i$ th row of  $Q^{(k)}$ , the first part of the theorem follows. The proof of the second part of the theorem is obtained by a similar argument.

Since every tree of distance  $k$  from the starting tree involves the replacement of a set of  $k$  branches, and since every possible replacement is represented by a row of  $Q^{(k)}$ , the following corollary is obvious.

#### Corollary 3-2

The number of trees of distance  $k$  from the starting tree of a graph is given by the trace of  $[Q Q^T]^{(k)}$ .

It should be noted that the same result is obtained by using the trace of  $[Q^T Q]^{(k)}$ . Since  $Q Q^T$  is a square matrix of order  $r$  and since  $Q^T Q$  is a square matrix of order  $\mu$ , the trace of the smaller matrix may be used in a given problem.

The following example illustrates Theorem 3-2 and its corollary. Example 3-3. For the graph of Figure 3-2, using adeg as the starting tree,  $Q$  is given by (3-8). Then

$$(Q Q^T)^{(1)} = \begin{matrix} & \begin{matrix} a & d & e & g & h \end{matrix} \\ \begin{matrix} a \\ d \\ e \\ g \\ h \end{matrix} & \begin{bmatrix} 2 & 2 & 0 & 2 & 0 \\ 2 & 4 & 2 & 2 & 0 \\ 0 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix}, \quad (3-21)$$

$$(Q Q^T)^{(2)} = \begin{matrix} & \begin{matrix} ad & ae & ag & ah & de & dg & dh & eg & eh & gh \end{matrix} \\ \begin{matrix} ad \\ ae \\ ag \\ ah \\ de \\ dg \\ dh \\ eg \\ eh \\ gh \end{matrix} & \left[ \begin{array}{c} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \right] \end{matrix} \quad (3-22)$$

or

$$(Q Q^T) (2) = \begin{matrix} & \begin{matrix} ad & ae & ag & ah & de & dg & dh & eg & eh & gh \end{matrix} \\ \begin{matrix} ad \\ ae \\ ag \\ ah \\ de \\ dg \\ dh \\ eg \\ eh \\ gh \end{matrix} & \left[ \begin{array}{cccccccccc} 4 & & & & & & & & & \\ & 6 & & & & & & & & \\ & & 2 & & & & & & & \\ & & & 2 & & & & & & \\ & & & & 8 & & & & & \\ & & & & & 8 & & & & \\ & & & & & & 4 & & & \\ & & & & & & & 8 & & \\ & & & & & & & & 2 & \\ & & & & & & & & & 2 \end{array} \right] \end{matrix}, \quad (3-23)$$

and

$$(Q Q^T) (3) = \begin{matrix} & \begin{matrix} ade & adg & adh & aeg & aeh & agh & deg & deh & dgh & egh \end{matrix} \\ \begin{matrix} ade \\ adg \\ adh \\ aeg \\ aeh \\ agh \\ deg \\ deh \\ dgh \\ egh \end{matrix} & \left[ \begin{array}{cccccccccc} 4 & & & & & & & & & \\ & 4 & & & & & & & & \\ & & 4 & & & & & & & \\ & & & 4 & & & & & & \\ & & & & 4 & & & & & \\ & & & & & 0 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & 4 & & \\ & & & & & & & & 4 & \\ & & & & & & & & & 4 \end{array} \right] \end{matrix}. \quad (3-24)$$

Therefore, the number of trees of distance 2 which may be obtained from (3-22) by replacing branches eg of t is given by

$$\det \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 8 \quad .$$

The number of trees of distance 2 from t is given by

$$\text{Trace } (Q Q^T)^{(2)} = 46 \quad .$$

Also, the number of trees of distance one from t is found to be

$$\text{Trace } (Q Q^T)^{(1)} = 13 \quad ,$$

and the number of trees of distance 3 from t is

$$\text{Trace } (Q Q^T)^{(3)} = 32 \quad .$$

These results can be verified by counting the nonzero elements in the compounds of Example 3-2.

Also,

$$Q^T Q = \begin{matrix} & \begin{matrix} b & c & f & i & j \end{matrix} \\ \begin{matrix} b \\ c \\ f \\ i \\ j \end{matrix} & \begin{bmatrix} 2 & 2 & -1 & -1 & -1 \\ 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & 3 & 3 \\ -1 & -1 & -1 & 3 & 3 \end{bmatrix} \end{matrix} \quad . \quad (3-25)$$

The number of trees which may be obtained from  $\text{adegh}$  by replacing pairs of branches of  $t$  by the chord pair  $ci$  is given by

$$\det \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 5 \quad .$$

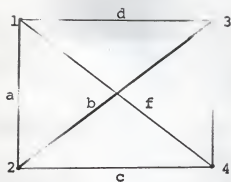


Fig. 3-1. Undirected Graph.

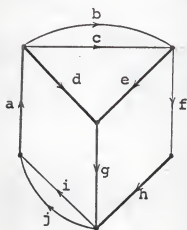


Fig. 3-2. Directed Graph.



#### IV. SOME SIMPLIFICATIONS OF CALCULATIONS

When a graph has a rather large number of edges and vertices, the labor involved in finding the trees by the compound method can be prohibitively large. In this chapter, a number of methods for reducing this labor are explored. It will be clear that some of these methods are mutually exclusive, while others can be used jointly. Some apply only to the compound method, while others can be applied to more classical tree-generating procedures. This chapter thus contains a collection of ideas relevant to the labor-reducing problem, rather than a description of a single optimizing procedure.

##### 1. Inspecting the Cotree for Circuits

In this section it is shown that it is often possible to find entire columns of zeros in the compounds by searching the cotree of the starting tree for circuits. Thus, many of the calculations required when a compound is generated directly from its definition are unnecessary.

First of all, suppose there is a pair of parallel edges in the cotree of a starting tree  $t$  of a graph  $G$  having rank  $r$  and nullity  $\mu$ . It follows from (3-2) that the entries of the two columns of  $Q$  corresponding to these two edges are identical. Thus, this pair of columns will result in a column of zeros in  $Q^{(2)}$ . Furthermore, any minors of order three of  $Q$  formed from a set of columns containing these two columns of  $Q$  as a subset must be zero. Thus, in  $Q^{(3)}$  there will be  $(\mu-2)$  columns of zeros. By the same argument the  $k$ th compound will have  $\binom{\mu-2}{k-2}$  columns

of zeros. Thus, if the cotree of  $t$  contains a pair of parallel edges, large numbers of entries in the compounds are known to be zero and need not be calculated. This can be seen by reviewing a previous example.

In Example 3-2, the compound method was illustrated for the graph of Figure 3-2 using  $adeqh$  as the starting tree. The cotree of  $adeqh$ ,  $bcfij$ , contains the parallel edge pairs  $bc$  and  $ij$ . In the second compound given by (3-9), columns  $bc$  and  $ij$  contain only zeros. In (3-10),  $Q^{(3)}$  has all zeros in every column which contains either  $bc$  or  $ij$  in its column label. Thus, in this case 80 entries in the last two nonzero compounds are easily found without calculations.

When there are more than two edges incident at the same pair of vertices, say  $q$  edges, then in  $Q^{(2)}$  there will be  $\binom{q}{2}$  columns of zeros, and in  $Q^{(3)}$  there will be  $\binom{q}{3} + \binom{q}{2} \binom{\mu-q}{1}$ . In general, in  $Q^{(k)}$ , there will be  $\sum_{i=2}^k \binom{q}{i} \binom{\mu-q}{k-i}$  columns of zeros for  $k < q$  and  $\sum_{i=2}^q \binom{q}{i} \binom{\mu-q}{k-i}$  for  $k \geq q$ .

Next, more generally, consider searching for circuits in the cotree. Let  $p$  chords constitute a circuit in the cotree, where  $p \leq l(r, \mu)$ . Then in the  $p$ th compound, a set of minors which corresponds to this set of chords will produce a column of zeros. By the reasoning used above, there will be  $\binom{\mu-p}{k-p}$  columns of zeros in the  $k$ th compound, where  $k \geq p$ .

## 2. Two Starting Trees

In this section a method is developed for finding the trees of a graph using two starting trees rather than one. The method applies only to certain of the graphs having the property  $\mu=r$ .

When the method is applicable, many of the required minors are of lower order than those needed when one tree is used, making the total labor less.

Consider the special case of a graph  $G$  such that  $\mu=r$ ,  $t_1$  is a tree of  $G$  and the cotree of  $t_1$ ,  $t_2$  is also a tree of  $G$ . The incidence matrix for  $G$  may be partitioned in the form

$$A = [A_{t_1} \quad A_{t_2}] \quad (4-1)$$

where the columns of  $A_{t_1}$  ( $A_{t_2}$ ) correspond to the branches of  $t_1$  ( $t_2$ ). Using (3-2), the f-cut-set matrix relative to  $t_1$  may be written in the form

$$Q_{f_1} = [U \ Q_1] = [U \ A_{t_1}^{-1} \ A_{t_2}] \quad (4-2)$$

Also, the f-cut-set matrix relative to  $t_2$  is

$$Q_{f_2} = [U \ Q_2] = [U \ A_{t_2}^{-1} \ A_{t_1}] \quad (4-3)$$

Thus  $Q_1$  and  $Q_2$  are related by  $Q_1 \ Q_2 = U$ . From Theorem 2-13, it follows that

$$[Q_1^{(r-k)}]_{p,q} = (-1)^s (\det Q_2)^{-1} [Q_2^{(k)}]_{n_k - q + 1, n_k - p + 1} \quad (4-4)$$

By Theorem 2-5,  $\det Q_2 = \pm 1$ . Then,  $(-1)^s (\det Q_2)^{-1}$  in (4-4) is merely a sign factor. Therefore, the higher order compounds of  $Q_1$  can be found from lower order compound of  $Q_2$ . Hence, some of the calculations for finding the compounds involve determinants of lower order than would be the case if one tree were used.

Next, it is necessary to determine how many compounds of

$Q_1$  and  $Q_2$  must be calculated so that no trees are located twice. When  $r$  is odd,  $\frac{r-1}{2}$  is even. Then compounds one through  $\frac{r-1}{2}$  of  $Q_1$  serve to locate one group of trees. Instead of calculating compounds  $(\frac{r-1}{2} + 1)$  through  $r$  of  $Q_1$  to find the remaining trees, compounds one through  $\frac{r-1}{2}$  of  $Q_2$  are used. Therefore, compounds one through  $\frac{r-1}{2}$  of both  $Q_1$  and  $Q_2$  will find all trees of  $G$ . Trees  $t_1$  and  $t_2$ , of course, must be included in the tree set. When  $r$  is even, the  $\frac{r}{2}$ th compound of  $Q_1$  and the  $\frac{r}{2}$ th compound of  $Q_2$  are identical. Therefore, one of these compounds can be neglected. Otherwise, the procedure is the same as when  $r$  is odd.

It is worthwhile to mention that when the field modulo 2 is used for undirected graphs, (4-4) reduces to the simple form:

$$[Q_1^{(r-k)}]_{p,q} = [Q_2^{(k)}]_{n_k - q + 1, n_k - p + 1} \quad (4-5)$$

The procedure of finding the trees of a graph from two starting trees is illustrated by the following example.

Example 4-1. Consider the graph shown in Figure 4-1. Let a starting tree be  $t_1 = bcdgh$ . The complement of  $t_1$ ,  $aefij = t_2$ , is also a tree. Then

$$Q_1^{(1)} = \begin{matrix} & \begin{matrix} a & e & f & i & j \end{matrix} \\ \begin{matrix} b \\ c \\ d \\ g \\ h \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

and

$$Q_2^{(1)} = \begin{matrix} & \begin{matrix} b & c & d & g & h \end{matrix} \\ \begin{matrix} a \\ e \\ f \\ i \\ j \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

Since the rank of  $G$  is five, compounds one through five of  $Q_1$  would be required to find all of the trees by the basic compound method. However, using (4-5), all trees other than  $t_1$  and  $t_2$  can be found using compounds one and  $\frac{5-1}{2} = 2$  of  $Q_1$  and  $Q_2$ . The second compounds are

$$Q_1^{(2)} = \begin{matrix} & \begin{matrix} ae & af & ai & aj & ef & ei & ej & fi & fj & ij \end{matrix} \\ \begin{matrix} bc \\ bd \\ bg \\ bh \\ cd \\ cg \\ ch \\ dg \\ dh \\ gh \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

and

$$Q_2^{(2)} = \begin{matrix} & bc & bd & bg & bh & cd & cg & ch & dg & dh & gh \\ \begin{matrix} ae \\ af \\ ai \\ aj \\ ef \\ ei \\ ej \\ fi \\ fj \\ ij \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix} .$$

As this example illustrates, the largest determinants which must be calculated are of only the second order. This saves much labor.

The application of Theorem 2-13 to graph theory problems is particularly powerful since it is relatively easy to calculate the inverses of the matrices which are encountered by machine. The idea of using two starting trees also greatly reduces the number calculations in the efficient Mayeda-Seshu tree-finding procedure, because the trees more distant from the starting tree tend to be more difficult to find in terms of this procedure.

### 3. Pivotal Condensation

Pivotal condensation is a well-known method of reducing a determinant of  $n$ th order to one of the  $(n-1)$ th order [22]. Its primary application is in making it possible to evaluate a given

determinant of any order by computation of only second order determinants. The method can be formulated in the following way.

Choose any nonzero element  $m_{i,j}$  in the determinant  $|M_n|$  as the pivot term. Select any element  $m_{i,k}$  which is in the same row as  $m_{i,j}$ , any element  $m_{q,j}$  which is in the same column as  $m_{i,j}$  and the element  $m_{q,k}$  which is the intersection of row  $q$  and column  $k$ . The elements  $m_{i,j}$ ,  $m_{i,k}$ ,  $m_{q,j}$  and  $m_{q,k}$  are then used to form a second order determinant, with  $m_{i,j}$  in the (1, 1) position and the others kept in their proper order. Form all such possible second order determinants with the pivot term as one of the elements. The original determinant  $|M_n|$  can then be expressed as an  $n$ -1th order determinant,  $|M_{n-1}|$ , using the second order minors of  $|M_n|$  as elements, and  $(-1)^{i+j}/m_{i,j}^{n-2}$  as a signed multiplying factor. By repeating this procedure, the value of a determinant of higher order can be computed by successively reducing the order of the determinant by one.

Consider the following fourth order determinant  $|M_4|$  for example. Let element,  $m_{2,3}$ , be the pivot term. Then

$$|M_4| = \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{vmatrix} \quad (4-6)$$

and

$$|M_4| = \frac{(-1)^{2+3}}{m_{2,3}} \begin{vmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{vmatrix} \quad (4-7)$$

in (4-7), where

$$b_{i,j} = \begin{vmatrix} m_{2,3} & m_{2,j} \\ m_{i,3} & m_{i,j} \end{vmatrix}, \text{ for } i = 1, \text{ and } j = 1, 2, \quad (4-8)$$

$$b_{i,j} = \begin{vmatrix} m_{2,3} & m_{2,j} \\ m_{i+1,3} & m_{i+1,j} \end{vmatrix}, \text{ for } i = 2, 3 \text{ and } j = 1, 2, \quad (4-9)$$

$$b_{i,j} = \begin{vmatrix} m_{2,3} & m_{2,j+1} \\ m_{i,3} & m_{i,j+1} \end{vmatrix}, \text{ for } i = 1, \text{ and } j = 3, \quad (4-10)$$

and

$$b_{i,j} = \begin{vmatrix} m_{2,3} & m_{2,j+1} \\ m_{i+1,3} & m_{i+1,j+1} \end{vmatrix}, \text{ for } i = 2, 3, \text{ and } j = 3. \quad (4-11)$$

If  $b_{1,2}$  is chosen as the pivot term for further reduction, then

$$|M_4| = \frac{(-1)(-1)^{1+2}}{m_{2,3} b_{1,2}} \begin{vmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{vmatrix} \quad (4-12)$$



where

$$c_{k,l} = \begin{vmatrix} b_{1,2} & b_{1,1} \\ b_{k+1,2} & b_{k+1,1} \end{vmatrix}, \text{ for } k = 1, 2, \text{ and } l = 1, \quad (4-13)$$

and

$$c_{k,l} = \begin{vmatrix} b_{1,2} & b_{1,l+1} \\ b_{k+1,2} & b_{k+1,l+1} \end{vmatrix}, \text{ for } k = 1, 2, \text{ and } l = 2. \quad (4-14)$$

From (4-12) the value of the determinant  $|M_4|$  can be obtained easily:

$$|M_4| = \frac{(-1)^2}{m_{2,3}^2 b_{1,2}} d_{1,1} = \frac{d_{1,1}}{m_{2,3}^2 b_{1,2}}, \quad (4-15)$$

where

$$d_{1,1} = \begin{vmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{vmatrix}.$$

An element from the determinant on the right side of (4-12) say  $c_{2,1}$ , can be expressed in terms of the elements of (4-6) as follows. From (4-13)

$$c_{2,1} = \begin{vmatrix} b_{1,2} & b_{1,1} \\ b_{3,2} & b_{3,1} \end{vmatrix}. \quad (4-16)$$

Substituting the proper expressions for  $b_{i,j}$  from (4-8) and (4-9) gives

$$\begin{aligned}
 c_{2,1} &= \begin{vmatrix} \begin{vmatrix} m_{2,3} & m_{2,2} \end{vmatrix} & \begin{vmatrix} m_{2,3} & m_{2,1} \end{vmatrix} \\ \begin{vmatrix} m_{1,3} & m_{1,2} \end{vmatrix} & \begin{vmatrix} m_{1,3} & m_{1,1} \end{vmatrix} \\ \begin{vmatrix} m_{2,3} & m_{2,2} \end{vmatrix} & \begin{vmatrix} m_{2,3} & m_{2,1} \end{vmatrix} \\ \begin{vmatrix} m_{4,3} & m_{4,2} \end{vmatrix} & \begin{vmatrix} m_{4,3} & m_{4,1} \end{vmatrix} \end{vmatrix} \\
 &= m_{2,3} \begin{vmatrix} m_{2,3} & m_{2,2} & m_{2,1} \\ m_{1,3} & m_{1,2} & m_{1,1} \\ m_{4,3} & m_{4,2} & m_{4,1} \end{vmatrix} \\
 &= (-1)^{1+3} m_{2,3} \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{4,1} & m_{4,2} & m_{4,3} \end{vmatrix}. \quad (4-17)
 \end{aligned}$$

Now, this method is applied to the problem of calculating the compounds of a matrix  $M$ . Since the third compound contains all possible third order minors of  $M$ , an element of the third compound can be expressed in terms of minors of order two from the second compound and a nonzero pivot term from the first compound. In general, the elements of the  $(n-1)$ th compound can be found from second minors of the  $n$ th compound and a nonzero pivot term from the  $(n-1)$ th compound.

Let  $x_{u_1, v_1}$  denote a general element of  $M^{(1)}$ , and let  $y_{u_1 u_2, v_1 v_2}$  denote a general element of  $M^{(2)}$  which, by definition, is a determinant of second order of  $M^{(1)}$ . That is

$$y_{u_1 u_2, v_1 v_2} = \begin{vmatrix} x_{u_1, v_1} & x_{u_1, v_2} \\ x_{u_2, v_1} & x_{u_2, v_2} \end{vmatrix}. \quad (4-18)$$

Now, consider calculating a general element of  $M^{(3)}$ , denoted as  $z_{u_1 u_2 u_3, v_1 v_2 v_3}$  by pivotal condensation. Since  $z_{u_1 u_2 u_3, v_1 v_2 v_3}$  is merely a minor of order three of  $M^{(1)}$ , by definition

$$z_{u_1 u_2 u_3, v_1 v_2 v_3} = \begin{vmatrix} x_{u_1, v_1} & x_{u_1, v_2} & x_{u_1, v_3} \\ x_{u_2, v_1} & x_{u_2, v_2} & x_{u_2, v_3} \\ x_{u_3, v_1} & x_{u_3, v_2} & x_{u_3, v_3} \end{vmatrix}. \quad (4-19)$$

Now if elements  $x_{u_1, v_1}$ ,  $x_{u_1, v_2}$ , and  $x_{u_1, v_3}$  are all zero,  $z_{u_1 u_2 u_3, v_1 v_2 v_3}$  evidently has the value zero and no calculations are necessary.

Thus, without loss of generality, select the first row subscript of the element of  $M^{(3)}$  and search for a nonzero element among the elements of  $M^{(1)}$  having this row subscript and one of the column subscripts of the element of  $M^{(3)}$  to use as a pivot term. If  $x_{u_1, v_3}$  of (4-19) is nonzero, for example, then by the pivotal condensation procedure,

$$z_{u_1 u_2 u_3, v_1 v_2 v_3} = \frac{(-1)^{1+3}}{x_{u_1, v_3}} \begin{vmatrix} x_{u_1, v_3} & x_{u_1, v_1} & x_{u_1, v_2} \\ x_{u_2, v_3} & x_{u_2, v_1} & x_{u_2, v_2} \\ x_{u_3, v_3} & x_{u_3, v_1} & x_{u_3, v_2} \end{vmatrix} \quad (4-20)$$

From (4-18)

$$\begin{vmatrix} x_{u_1, v_3} & x_{u_1, v_1} \\ x_{u_2, v_3} & x_{u_2, v_1} \end{vmatrix} = - \begin{vmatrix} x_{u_1, v_1} & x_{u_1, v_3} \\ x_{u_2, v_1} & x_{u_2, v_3} \end{vmatrix} = - y_{u_1 u_2, v_1 v_3} ,$$

and

$$\begin{vmatrix} x_{u_1, v_3} & x_{u_1, v_2} \\ x_{u_3, v_3} & x_{u_3, v_2} \end{vmatrix} = - \begin{vmatrix} x_{u_1, v_2} & x_{u_1, v_3} \\ x_{u_3, v_2} & x_{u_3, v_3} \end{vmatrix} = - y_{u_1 u_3, v_1 v_3} ,$$

and similarly for the other terms. Thus (4-20) becomes

$$z_{u_1 u_2 u_3, v_1 v_2 v_3} = \frac{(-1)^{1+3} (-1)^2}{x_{u_1, v_3}} \begin{vmatrix} y_{u_1 u_2, v_1 v_3} & y_{u_1 u_2, v_2 v_3} \\ y_{u_1 u_3, v_1 v_3} & y_{u_1 u_3, v_2 v_3} \end{vmatrix} , \quad (4-21)$$

where (4-21) expresses a general element of  $M^{(3)}$  in terms of a minor of order two from  $M^{(2)}$  and a pivot element from  $M^{(1)}$ . The sign,  $(-1)^{1+3} (-1)^2 = (-1)^6 = 1$ , in (4-21) which comes from the choice of the pivot term  $x_{u_1, v_3}$  is positive. Further thought reveals that when any element in the first row of (4-19) is chosen as the pivot term, say  $x_{u_1, v_i}$ , then the sign in (4-21) will be  $(-1)^{1+i} (-1)^{i-1} = (-1)^{2i}$ . That is the sign is always positive. Therefore, the element  $z_{u_1 u_2 u_3, v_1 v_2 v_3}$  of  $M^{(3)}$  can be found by picking any nonzero element from the row  $u_1$  of  $M^{(1)}$  as the pivot term and by calculating the corresponding second order determinant of  $M^{(2)}$ . The sign is always positive.

Next, consider a general element  $w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4}$  of  $M^{(4)}$ .

By definition,

$$w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4} = \begin{vmatrix} x_{u_1, v_1} & x_{u_1, v_2} & x_{u_1, v_3} & x_{u_1, v_4} \\ x_{u_2, v_1} & x_{u_2, v_2} & x_{u_2, v_3} & x_{u_2, v_4} \\ x_{u_3, v_1} & x_{u_3, v_2} & x_{u_3, v_3} & x_{u_3, v_4} \\ x_{u_4, v_1} & x_{u_4, v_2} & x_{u_4, v_3} & x_{u_4, v_4} \end{vmatrix}. \quad (4-22)$$

Pick any nonzero element in row  $u_1$  of  $M^{(1)}$  as a pivot term, say

$x_{u_1, v_2}$ . Then

$$w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4} = \frac{(-1)^{1+2}}{x_{u_1, v_2}^2}.$$

$$\begin{vmatrix} \begin{vmatrix} x_{u_1, v_2} & x_{u_1, v_1} \end{vmatrix} & \begin{vmatrix} x_{u_1, v_2} & x_{u_1, v_3} \end{vmatrix} & \begin{vmatrix} x_{u_1, v_2} & x_{u_1, v_4} \end{vmatrix} \\ \begin{vmatrix} x_{u_2, v_2} & x_{u_2, v_1} \end{vmatrix} & \begin{vmatrix} x_{u_2, v_2} & x_{u_2, v_3} \end{vmatrix} & \begin{vmatrix} x_{u_2, v_2} & x_{u_2, v_4} \end{vmatrix} \\ \begin{vmatrix} x_{u_3, v_2} & x_{u_3, v_1} \end{vmatrix} & \begin{vmatrix} x_{u_3, v_2} & x_{u_3, v_3} \end{vmatrix} & \begin{vmatrix} x_{u_3, v_2} & x_{u_3, v_4} \end{vmatrix} \\ \begin{vmatrix} x_{u_4, v_2} & x_{u_4, v_1} \end{vmatrix} & \begin{vmatrix} x_{u_4, v_2} & x_{u_4, v_3} \end{vmatrix} & \begin{vmatrix} x_{u_4, v_2} & x_{u_4, v_4} \end{vmatrix} \end{vmatrix}. \quad (4-23)$$

But since

$$\begin{vmatrix} x_{u_1, v_2} & x_{u_1, v_1} \\ x_{u_2, v_2} & x_{u_2, v_1} \end{vmatrix} = - \begin{vmatrix} x_{u_1, v_1} & x_{u_1, v_2} \\ x_{u_2, v_1} & x_{u_2, v_2} \end{vmatrix} = - y_{u_1 u_2, v_1 v_2} ,$$

(4-23) becomes

$$w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4} = \frac{(-1)^{1+2} (-1)^1}{x_{u_1, v_2}} \begin{vmatrix} y_{u_1 u_2, v_1 v_2} & y_{u_1 u_2, v_2 v_3} & y_{u_1 u_2, v_2 v_4} \\ y_{u_1 u_3, v_1 v_2} & y_{u_1 u_3, v_2 v_3} & y_{u_1 u_3, v_2 v_4} \\ y_{u_1 u_4, v_1 v_2} & y_{u_1 u_4, v_2 v_3} & y_{u_1 u_4, v_2 v_4} \end{vmatrix} .$$

(4-24)

The same argument for the sign in (4-21) holds for the sign of (4-24). That is, the sign is positive for any choice of pivot element. Now, (4-24) is reduced to a third order determinant, repeating the procedure (4-19) through (4-21). If  $y_{u_1 u_2, v_2 v_4}$  is nonzero, for example,

$$w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4} = \frac{(-1)^{1+3}}{(x_{u_1, v_2}^2) (y_{u_1 u_2, v_2 v_4})} .$$

$$\begin{vmatrix} y_{u_1 u_2, v_2 v_4} & y_{u_1 u_2, v_1 v_2} \\ y_{u_1 u_3, v_2 v_4} & y_{u_1 u_3, v_1 v_2} \\ y_{u_1 u_4, v_2 v_4} & y_{u_1 u_4, v_1 v_2} \end{vmatrix} \begin{vmatrix} y_{u_1 u_2, v_2 v_4} & y_{u_1 u_2, v_2 v_3} \\ y_{u_1 u_3, v_2 v_4} & y_{u_1 u_3, v_2 v_3} \\ y_{u_1 u_4, v_2 v_4} & y_{u_1 u_4, v_2 v_3} \end{vmatrix} . \quad (4-25)$$

By (4-21) and by the fact that

$$\begin{vmatrix} y_{u_1 u_2, v_2 v_4} & y_{u_1 u_2, v_1 v_2} \\ y_{u_1 u_3, v_2 v_4} & y_{u_1 u_3, v_1 v_2} \end{vmatrix} = - \begin{vmatrix} y_{u_1 u_2, v_1 v_2} & y_{u_1 u_2, v_2 v_4} \\ y_{u_1 u_3, v_1 v_2} & y_{u_1 u_3, v_2 v_4} \end{vmatrix}$$

$$= (-1)^{1+2} (-1)^1 x_{u_1, v_2} z_{u_1 u_2 u_3, v_1 v_2 v_4},$$

(4-25) becomes

$$w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4} = \frac{(-1)^{1+3} (-1)^{1+2} (-1)}{y_{u_1 u_2, v_2 v_4}}.$$

$$\begin{vmatrix} z_{u_1 u_2 u_3, v_1 v_2 v_4} & z_{u_1 u_2 u_3, v_2 v_3 v_4} \\ z_{u_1 u_2 u_4, v_1 v_2 v_4} & z_{u_1 u_2 u_4, v_2 v_3 v_4} \end{vmatrix} \quad (4-26)$$

which is the desired result.

The argument concerning the sign  $(-1)^{1+3} (-1)^{1+2} (-1) = (-1)^8 = 1$  of (4-26) is the same as discussed previously. Thus, the element  $w_{u_1 u_2 u_3 u_4, v_1 v_2 v_3 v_4}$  of  $M^{(4)}$  can be found by choosing a nonzero element in the  $u_1 u_2$  row of  $M^{(2)}$  as the pivot term and by then calculating the appropriate second order minor of  $M^{(3)}$ .

The procedure holds for finding an element of  $M^{(k)}$ , in general, from a nonzero element of  $M^{(k-2)}$  and the corresponding second order minor from  $M^{(k-1)}$ .

Since a nonzero entry of  $Q$  as well as its compounds is either  $+1$  or  $-1$ , a selected pivot term in  $Q^{(k-2)}$  merely affects the sign of the corresponding element of  $Q^{(k)}$ . It is noted that when the field modulo 2 is used, the pivot term is always 1.

#### 4. The Central Tree

Deo recently introduced the concept of a central tree and demonstrated its usefulness in reducing the labor involved in listing the trees of a connected graph [16]. Deo's definition of the central tree is stated as follows.

A central tree is a tree  $t_0$  in  $G$  such that the rank  $r$  of its complement  $\bar{t}_0$  is a minimum. That is,  $\text{rank}(\bar{t}_0) \leq \text{rank}(\bar{t}_i)$ , for every tree  $t_i$  belonging to  $G$ .

Here the central tree is viewed in a new context which suggests an additional reduction in labor.

Let

$$Q_f = [U \quad Q] \quad (4-27)$$

be the  $f$ -cut-set matrix of a connected graph  $G$  of rank  $r$  and nullity  $\mu$  with respect to a starting tree  $t$ . It was shown in the preceding chapter that there exists a one-to-one correspondence between the trees of distance  $k$  from  $t$  and the nonzero elements of  $Q^{(k)}$ ; and that all trees of  $G$  may be found from compounds one through  $l(r, \mu)$  of  $Q$ . Now if  $Q$  is of rank  $p$ , it follows from the definition of rank that

$$Q^{(p+w)} = 0 \quad (4-28)$$



for all integers  $w > 0$ . Thus if  $t$  is chosen such that  $Q$  is of minimum possible rank,  $p_m$ , then no tree is of distance greater than  $p_m$  from  $t$ . Such a starting tree satisfies Deo's definition of a central tree because  $Q$  is related by a nonsingular transformation to the reduced incidence matrix of the complement of  $t$  as shown in (3-2).

Let  $t_0$  be a central tree of  $G$  such that  $Q$  is of rank  $p_m < l(r, \mu)$ . As discussed previously, trees of distance greater than  $p_m$  are known to be zero; and all trees are located using only the first  $p_m$  compounds. This type of labor reduction applies not only to the compound method, but to any method in which the trees are located recursively in groups according to distance from the starting tree. Deo has pointed out that considerable labor is saved by proper choice of the starting tree in a procedure given by Hakimi [8]. This also applies to the relatively efficient Mayeda-Seshu procedure [13].

The compound viewpoint reveals an additional computational advantage of starting with the central tree. By a well-known theorem of matrix theory (Theorem 5.22.3 by Hohn [23]), all non-zero rows of  $Q^{(p_m)}$  must have nonzero elements in the same columns. Thus, if one nonzero row is found, one need only find one nonzero entry in this row and then search the corresponding column of  $Q^{(p_m)}$  for nonzero entries. Whenever a zero is found in a particular row, the entire row is zero. Whenever a nonzero entry is found, nonzero

entries necessarily appear in the same columns as in the starting row. Thus  $Q^{(p_m)}$  is completely determined if one nonzero row is known by examining  $\binom{r}{p_m} - 1$  minors of order  $p_m$  of  $Q$ .

The following example illustrates how this result may be used in conjunction with the Mayeda-Seshu tree-generating procedure.

Example 4-2. Figure 4-2 shows a graph given by Deo [16] and the corresponding  $Q$ , mod 2, for  $t_0 = abci j$ . Since  $Q$  is of rank  $p_m = 4$ , the simplification applies to

$$Q^{(4)} = \begin{matrix} & \begin{matrix} defg & defh & deg h & dfgh & efgh \end{matrix} \\ \begin{matrix} abci \\ abcj \\ abij \\ acij \\ bcij \end{matrix} & \left[ \begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array} \right] \end{matrix} \quad (4-29)$$

in which the row proportionality is obvious. Clearly, one need only calculate one nonzero row, say row  $abci$ , and then calculate the entries of one column, say  $defg$ , in order to completely specify  $Q^{(4)}$ . When high nullity graphs are considered,  $Q^{(4)}$  may have many columns, and the reduction in labor can be significant.

Suppose that by the Mayeda-Seshu procedure one finds the tree set

$$\tau_{abci} = \{jdefg, jdfgh, jefgh\}. \quad (4-30)$$

These trees correspond to nonzero elements in columns  $defg$ ,  $dfgh$ , and  $efgh$  of row  $abci$  of  $Q^{(4)}$ . All other trees of distance four

must be contained in  $T^{abcj}$ ,  $T^{abij}$ , and  $T^{bcij}$ ; which correspond to the remaining rows of  $Q^{(4)}$ . To determine which of these sets are nonempty, examine  $idefg$ ,  $cdefg$ ,  $bdefg$ ,  $adefg$ . All are trees with exception of  $cdefg$ . Or alternatively, calculate the following minors of  $Q$ :

$$\begin{array}{c} \begin{array}{ccccc} & d & e & f & g \\ \begin{array}{c} a \\ b \\ c \\ j \end{array} & \left| \begin{array}{cccc} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \right| \end{array} = 1 ,$$

$$\begin{array}{c} \begin{array}{ccccc} & d & e & f & g \\ \begin{array}{c} a \\ b \\ i \\ j \end{array} & \left| \begin{array}{cccc} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right| \end{array} = 0 ,$$

$$\begin{array}{c} \begin{array}{ccccc} & d & e & f & g \\ \begin{array}{c} a \\ c \\ i \\ j \end{array} & \left| \begin{array}{cccc} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right| \end{array} = 1 ,$$

$$\begin{array}{c} \begin{array}{ccccc} & d & e & f & g \\ \begin{array}{c} b \\ c \\ i \\ j \end{array} & \left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right| \end{array} = 1 .$$

By either procedure, the conclusion is that

$$T^{abcj} = idefg, idfgh, iefgh,$$

$$T^{abij} = \emptyset$$

$$T^{acij} = bdefg, bdfgh, befgh,$$

$$T^{bcij} = adefg, adfgh, aefgh,$$

are the remaining trees of distance four.

The reduction in labor required to generate the trees of distance  $p_m$  is significant, particularly when there are many trees of distance  $p_m - 1$  which must be considered in finding the trees of distance  $p_m$  by the Mayeda-Seshu procedure.

The principle difficulty with the central tree idea is that there is no known way of finding the central tree for a general graph [16]. Some progress toward this goal is, however, reported in the following paragraphs.

At present, the starting tree is calculated by computer [13] with no guidelines built into the program to optimize this choice in any way. Lacking guidelines, the computer may possibly select a tree such that the rank of the corresponding  $Q$  is the maximum possible rank. The following theorem eliminates this possibility for simple graphs such that  $\mu \geq r$ .

#### Theorem 4-1

Let  $G$  be a connected, simple graph of rank  $r$ . It is always possible to find a tree  $t_i$  such that the complement of  $t_i$  is of rank  $r-1$ .

Proof: Let  $v_i$  be any vertex of  $G$ . The set of edges incident at  $v_i$  contains no circuits because  $G$  is simple. By Theorem 2-1, the set of edges incident at  $v_i$  can be made part of a tree. Let  $t_i$  be any tree of  $G$  containing the edges incident at  $v_i$  in  $G$ . Then the complement of  $t_i$  has at most  $r$  vertices in a connected part and therefore has a maximum rank of  $r-1$ . This completes the proof.

Theorem 4-1 at first appears trivial and of limited value, however further examination reveals that it is quite useful.

When  $G$  is such that  $r \leq \mu$ , the maximum possible rank of the appropriate matrix  $Q$  is  $r$ . Theorem 4-1 ensures that a tree can be found such that the rank of  $Q$  is  $r-1$ . Thus the preceding discussion concerning the proportionality of the rows of a compound and the resulting reduction of labor applies to  $Q^{(r-1)}$  in this case.

Theorem 4-1 also leads directly to a method for finding the central tree of a complete graph, as shown in the following theorem.

#### Theorem 4-2

Let  $t_i$  be the tree of a complete graph  $G^*$  consisting of all of the edges incident at any vertex  $v_i$  of  $G^*$ . Then  $t_i$  is a central tree of  $G^*$ .

Proof:

Let  $\bar{t}_i$  denote the complement of  $t_i$  with respect to  $G^*$ . Since  $G^*$  is a complete graph,  $\bar{t}_i$  consists of a connected subgraph containing  $r$  of the vertices of  $G^*$ , plus the isolated vertex  $v_i$ . Thus  $\bar{t}_i$  is of rank  $r-1$ . To prove that  $t_i$  is a central tree it is necessary to show that there does not exist a tree  $t$  of  $G^*$  such that the rank of the complement,  $\bar{t}$ , of  $t$  is less than  $r-1$ . Assume such a tree  $t$  exists. Since  $\bar{t}$  is of rank less than  $r-1$ ,  $\bar{t}$  consists of more than two vertex disjoint subgraphs, each of which is itself connected. First, suppose  $\bar{t}$  consists of the three connected subgraphs  $G_1^*$ ,  $G_2^*$ , and  $G_3^*$  as shown in Figure 4-3 (in this case, the rank of  $\bar{t}$  would be  $r-2$ ). Since  $G^*$  is a complete graph, there

exists in  $G^*$  an edge  $e_1$  incident at some vertex  $v_1$  in  $G_1^*$  and some vertex  $v_2$  in  $G_2^*$ . Similarly, there exist edges  $e_2$  and  $e_3$  in  $G^*$  incident at vertex pairs  $(v_2, v_3)$  and  $(v_3, v_1)$  respectively, where  $v_3$  is a vertex of  $G_3^*$ . Edges  $e_1$ ,  $e_2$ , and  $e_3$  are not in  $\bar{t}$ , therefore they are branches of  $t$ . But this is a contradiction, since  $t$  contains no circuit.

For any tree  $t$ , such that  $\bar{t}$  contains more than three isolated subgraphs, the preceding argument applies to any three of the subgraphs. Therefore, there is no tree of  $G^*$  having a complement of rank less than  $r-1$ , and the theorem is proved.

The following development extends the applicability of Theorem 4-1 to the case  $\mu < r$  when some additional restrictions are placed upon the graph.

When a simple graph  $G$  is of nullity  $\mu$  less than the rank  $r$ , the maximum possible rank of the complement of a starting tree depends upon  $l(r, \mu) = \mu$ . Thus Theorem 4-1 does not appear to offer any computational advantage with this kind of graph. However, when the graph  $G$  is planar, by Theorem 2-3, the dual graph  $G'$  of  $G$  always exists. In addition to the restrictions of Theorem 4-1, let  $G$  contain no series edges. Then the dual graph  $G'$  contains no parallel edges, and  $G'$  is a simple graph. Thus Theorem 4-1 can be applied to  $G'$ . The following example illustrates the usefulness of Theorem 4-1 associated with the duality concept.

Example 4-3. Use Theorem 4-1 to reduce the number of nonzero compounds of  $Q$  to less than the maximum possible number for the graph  $G$  of Figure 4-4(a).

For any starting tree, the order of  $Q$  is  $7 \times 5$ . Using the idea of Theorem 4-1, choose  $t_1$  such that edges  $afg$  belong to  $t_1$  and vertex 3 is isolated. Then the rank of  $Q$  will not exceed  $r - 1 = 6$ . For example, let  $t_1 = afgbhjd$ . For this tree, the rank of  $Q$  is 5, and 5 compounds are needed using this tree. Thus Theorem 4-1 seems to be of little value in this case. It is noted that another choice of the starting tree may have a corresponding matrix  $Q$  of rank less than 5. However, finding such a tree requires further labor.

Now, consider the dual graph  $G'$  of  $G$  as shown in Figure 4-4(c). For  $G'$ , the nullity  $\mu' = 7$  exceeds the rank  $r' = 5$ , and Theorem 4-1 applies directly. Choose  $t'_1$  such that edges  $d'e'j'k'$  are branches of  $t'_1$ . Then the rank of any matrix  $Q'$  does not exceed  $r' - 1 = 4$ . If  $b'$  is chosen as the other branch of  $t'_1$ , then the rank of the corresponding  $Q'$  is 4. Thus, all trees of  $G'$  may be found using only the first four (not five) compounds of  $Q'$ , and the labor reductions previously mentioned apply. But the cotrees of  $G'$  are in one-to-one correspondence with the trees of  $G$ . Thus the trees of  $G$  then can be found using fewer calculations by working with the dual graph.

It is also possible to use the dual graph  $G'$  merely to find a starting tree of  $G$ , and to then calculate the trees of  $G$  directly. As before, Theorem 4-1 is used to find a starting tree of  $G'$ , say  $t'_1 = d'e'j'k'b'$ . These edges correspond to the cotree  $\bar{t}_1 = dejkb$ , of a tree,  $t_1 = acfghil$ , of  $G$ . Figure 4-4(d) shows  $\bar{t}_1$



is of rank 4. Thus the trees of  $G$  may be found using only the first four compounds of matrix  $Q_1$ , which is found from  $t_1$ .

According to Deo's definition, the problem of finding a central tree of a general connected graph  $G$  amounts to finding a set of edges such that the complement of these edges with respect to  $G$  contains a maximum number of unconnected pieces. Here, an isolated vertex is considered an unconnected piece of a graph. It is useful to restrict the problem of finding the central tree to simple graphs for two reasons. First, as shown in the proof of Theorem 4-1, this makes it possible to use Theorem 2-1 as a powerful mathematical tool which lends a great deal of clarity to the reasoning. Second, the restricted nature of simple graphs is more obvious than real. Suppose, for example, it is desired to find the set of all trees of a graph  $G$  containing parallel edges  $e_1$  and  $e_2$  as illustrated in Figure 4-5(a). Let  $e_1$  and  $e_2$  be replaced by a single equivalent edge,  $e_3$ , to form a new graph  $G_s$ , as shown in Figure 4-5(b). The set of all trees of  $G_s$  may be divided into two-mutually-exclusive tree sets: the set of all trees which contain  $e_3$ , and the set of all trees which do not contain  $e_3$ . Clearly, each member of the latter is a tree of  $G$ . Also, let the former be used to generate two new sets: one formed by replacing  $e_3$  by  $e_1$ , the other by replacing  $e_3$  by  $e_2$ . These three tree sets are then the set of all trees of  $G$ . This line of thought is easily extended to more general cases involving  $m$  sets of parallel edges with varying numbers of edges in each set. The point is that the trees of a general graph  $G$  can be found in a rather straightforward manner once the trees of a related simple graph  $G_s$  are found.



In studying many special cases, it has always been found that the central trees are those trees having complements which contain a maximum number of isolated vertices. The algorithm which follows is an attempt to systematically maximize the number of isolated vertices in the complement. In all special cases examined, this algorithm has generated the central tree. However, it has not been possible to either prove or disprove the hypothesis that this algorithm always results in a central tree. The algorithm is therefore presented as a systematic method of finding a starting tree which is obviously preferable to a "random" choice. The algorithm is restricted to simple graphs.

#### Algorithm

1. Let  $v_1$  be one of the vertices of lowest degree of  $G$ . If the set,  $E_1$ , of edges incident at  $v_1$  constitutes a tree  $t_1$ , then  $t_1$  is the desired tree and the procedure terminates. (This occurs in the special case of a complete graph, for example.)
2. If the set  $E_1$  is not a tree of  $G$ , examine the vertices of the same degree as  $v_1$ , or if there are none, the vertices of next higher degree. In this examination, vertices having edges connected to  $v_1$  are given precedence. This results in a new vertex  $v_2$  with a set of edges  $E_2$  incident to it. Let  $U$  in this context denote the set theoretic union. If  $E_1 \cup E_2$  contains a circuit, then a new vertex is chosen by the same criteria to replace  $v_2$ . If the union of edges  $E_1 \cup E_2$  contains no circuit, the edges  $E_1 \cup E_2$  can be made part of the desired tree. A search is then started for a new vertex  $v_3$ , treating  $E_1 \cup E_2$  in the same manner as  $E_1$  was treated in this step.

3. Continuing this procedure must eventually lead to one of two possibilities. (a) A tree is found, terminating the procedure. (b) A point is reached such that the union of the edges incident at every remaining vertex and the previously found edge set,  $E_1 \cup E_2 \cup \dots \cup E_n$ , contains circuits. Select the remaining edges to complete the tree, if possible choosing edges such that the complement of  $E_1 \cup E_2 \cup \dots \cup E_n$  is split into unconnected pieces by removal of these edges.

The algorithm is illustrated by the following example.

Example 4-4. For the graph of Figure 4-6(a), vertices 1, 4, 5, and 6 are of degree three, and vertices 2, 3, 7, 8, 9, and 10 are of degree four. Since the graph has ten vertices, every tree has nine edges. Let vertex 1 be chosen as the starting point. Set  $E_1 = abc$  is not a tree. The edges incident at 6 form a circuit with  $E_1$ , so the next vertex must be chosen from (4, 5). Choose vertex 5, giving  $E_2 = opq$ . The set  $E_1 \cup E_2 = abcopq$  is not a tree. Three more edges are required for a tree. The edges incident at vertex 4 form a circuit when combined with  $E_1 \cup E_2$ . Since every other vertex is of degree four, the situation described in 3(b) has occurred. Figure 4-6(b) shows the complement of  $E_1 \cup E_2$ . Many combinations of three edges from this graph can be combined with  $E_1 \cup E_2$  to form a tree of the original graph. For example,  $(rij) \cup E_1 \cup E_2$  is a tree of the graph. However removing  $(rij)$  from the graph of Figure 4-6(b) leaves a connected graph. On the other hand,  $(rih) \cup E_1 \cup E_2$  is a tree of the original graph; and removing  $(rih)$  from the graph of Figure 4-6(b) leaves two unconnected

pieces. Thus rihabcpq is the final result. Other results are also possible. For example, had vertex 4 been chosen rather than vertex 5, the algorithm would have given the tree abcgihlnq. Both trees have cotrees of rank six.

## 5. Geometrical Interpretation Using Tree Graph

In this section some of the previous results are given a geometrical interpretation in terms of Cummins' tree graph [17]. The tree graph is defined as follows.

The tree graph  $\underline{G}$  corresponding to a connected graph  $G$  is such that the vertices of  $\underline{G}$  are in one-to-one correspondence with the trees of  $G$ . An edge is incident at a vertex pair of  $\underline{G}$  if and only if the corresponding tree pair of  $G$  are of distance one.

Let  $G$  denote a connected graph of rank  $r$  and nullity  $\mu$ , and let  $\underline{G}$  denote the corresponding tree graph. Symbols  $v_i$  and  $t_i$  denote a vertex of  $\underline{G}$  and the corresponding tree of  $G$ , respectively.

### Theorem 4-3

The tree graph  $\underline{G}$  is connected.

Proof: Let  $v_0$  and  $v_k$  be any two vertices of  $\underline{G}$ . There is some nonzero distance  $k$  between  $t_0$  and  $t_k$ . Let  $Q$  be obtained, as before, from the  $f$ -cut-set matrix of  $G$  with respect to  $t_0$ . There exists a nonzero minor  $M_k$  of  $Q$  of order  $k$ , with columns (rows) corresponding to the  $k$  branches of  $t_k$  ( $t_0$ ) not in  $t_0$  ( $t_k$ ). Since  $M_k \neq 0$ , not every minor of order  $k-1$  contained in  $M_k$  is zero. Let  $M_{k-1}$  denote such a nonzero minor. Minor  $M_{k-1}$  corresponds to a tree  $t_{k-1}$  of distance  $k-1$  from  $t_0$ . Furthermore,  $t_{k-1}$  can be

formed by removing some branch of  $t_k$  not common to  $t_0$  and  $t_k$  and by then adding a branch of  $t_0$  which was not a branch of  $t_k$ . The column (row) of  $M_k$  not contained in  $M_{k-1}$  corresponds to the removed (added) edge. Since  $t_{k-1}$  and  $t_k$  differ by exactly one branch, there is an edge in  $\underline{G}$  incident at both  $v_{k-1}$  and  $v_k$ . Similarly, there exists a tree  $t_{k-2}$  of  $G$  of distance  $k-2$  from  $t_0$ , and an edge in  $\underline{G}$  incident at both  $v_{k-2}$  and  $v_{k-1}$ . Continuing this argument establishes the existence of a path in  $\underline{G}$  (of length  $k$ ) having  $v_0$  and  $v_k$  as endpoints. This completes the proof.

#### Theorem 4-4

The distance between two vertices of  $\underline{G}$  which correspond to a tree-pair of  $G$  of distance  $k$ , is  $k$ . The distance between two trees of  $G$  corresponding to a vertex pair of  $G$  of distance  $k$ , is  $k$ .

Proof: Let  $t_0$  and  $t_k$  be trees of  $G$  of distance  $k$ . From the proof of Theorem 4-3, there exists a path of length  $k$  in  $\underline{G}$  having  $v_0$  and  $v_k$  as endpoints. Thus, the distance between  $v_0$  and  $v_k$  does not exceed  $k$ . Now assume  $v_0$  and  $v_k$  are of distance  $d < k$ . Then, there exists a path  $p_d$  of length  $d$  in  $\underline{G}$  having  $v_0$  and  $v_k$  as endpoints. Since each edge of  $p_d$  is incident at a vertex pair representing a pair of trees, differing by one edge,  $p_d$  implies that  $t_0$  and  $t_k$  differ by at most  $d$  edges, which contradicts the hypothesis.

Let  $v_0$  and  $v_k$  be a vertex pair of  $\underline{G}$  of distance  $k$ . The path of length  $k$  in  $\underline{G}$  with  $v_0$  and  $v_k$  as endpoints ensures that the distance between  $t_0$  and  $t_k$  does not exceed  $k$ . Now assume  $t_0$  and  $t_k$  are of distance  $d < k$ . By the proof of Theorem 4-3, there is a

path of length  $d$  in  $\underline{G}$  having  $v_o$  and  $v_k$  as endpoints, which contradicts the hypothesis. This completes the proof.

The following corollaries follow from Theorem 4-4 and the relationships established previously.

Corollary 4-4-1 The diameter of  $\underline{G}$  does not exceed  $l(\mu, r)$ .

Corollary 4-4-2 A tree of  $G$  such that the corresponding  $Q$  is of minimum rank corresponds to a center of  $\underline{G}$ . That is, a central tree of  $G$  corresponds to a center vertex of  $\underline{G}$ .

Corollary 4-4-3 The highest compound which must be calculated in order to find all of the trees of  $G$  is  $k_{\max}$ , where  $k_{\max}$  is the radius of  $\underline{G}$ .

Finally, it is noted that the non-uniqueness of the central tree, as discussed by Deo, could be inferred from the well-known fact that the center of a graph is not unique [20].

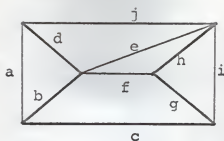
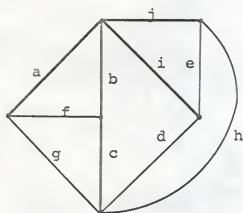


Fig. 4-1.



$$Q = \begin{matrix} & \begin{matrix} d & e & f & g & h \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ i \\ j \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{matrix}, P_m = 4.$$

Fig. 4-2.

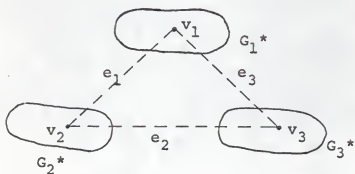
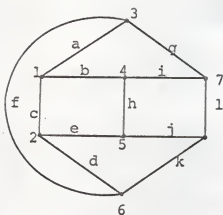
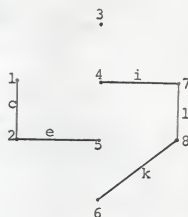


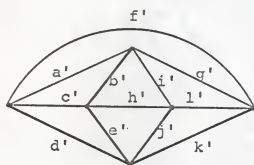
Fig. 4-3.



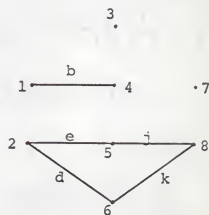
(a) Graph G



(b) Cotree of afgbhjd



(c) Graph G'



(d) Cotree of afgchil

Fig. 4-4.

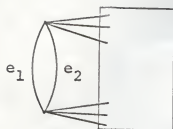
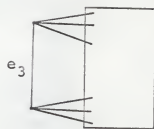
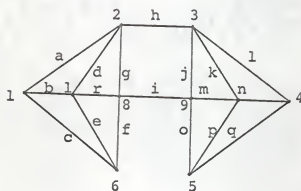
(a) Graph  $G$ (b) Graph  $G_s$ 

Fig. 4-5.



(a) Simple Graph

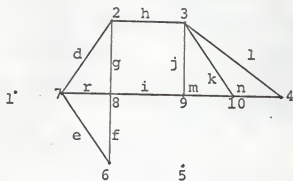
(b) Complement of  $E_1UE_2$ 

Fig. 4-6.



## V. FINDING COMPLETE TREES OF ACTIVE NETWORKS

In this chapter, the compound method is applied to the problem of finding the signed complete trees of graph pairs associated with active networks. Active networks are frequently analyzed using a graph pair, a voltage graph  $G_v$  and a current graph  $G_i$ , constructed upon identical sets of vertices. Every passive network element is represented by an edge, called an ordinary edge, which appears in both graphs incident at corresponding vertex pairs. Every active network element is represented by a pair of edges called an active edge pair, both weighted by the mutual admittance of the active element. One edge of an active edge pair appears only in the voltage graph, the other only in the current graph; and the edges appear incident at non-corresponding node pairs in the two graphs [5]. A complete tree of a graph pair is a set of  $r$  edges which is a tree of both the voltage graph and the current graph.

Associated with every complete tree  $t_j$ , containing an active edge, is an algebraic multiplier  $\epsilon_j = \pm 1$ , called the sign of  $t_j$ . The sign associated with every other complete tree is  $+1$ . If  $A_v$  and  $A_i$  are, respectively, the incidence matrices associated with  $G_v$  and  $G_i$ , having rows and columns in identical vertex and edge order, then the sign associated with  $t_j$  is expressed as

$$\epsilon_j = (\det A_{vt_j}) (\det A_{it_j}) \quad (5-1)$$

where the determinants on the right side of (5-1) are majors

of  $A_v$  and  $A_i$ , respectively, with columns corresponding to the edges of  $t_j$ .

Let  $t$  be a given complete tree of a graph pair, and let

$$\begin{aligned} A_v &= [A_{vt} \quad A_{vc}] \\ A_i &= [A_{it} \quad A_{ic}] \end{aligned} \quad (5-2)$$

be the partitioned incidence matrices of the graph pair with columns corresponding to branches appearing first. These matrices are written in identical vertex and edge order. Matrices  $Q_v$  and  $Q_i$  are the  $f$ -cut-set matrices

$$Q_{fv} = [U \quad Q_v] = [U \quad A_{vt}^{-1} \quad A_{vc}]$$

and

$$Q_{fi} = [U \quad Q_i] = [U \quad A_{it}^{-1} \quad A_{ic}] \quad (5-3)$$

It is clear that there must be a one-to-one correspondence between the complete trees of distance  $k$  from  $t$  and the nonzero entries in both  $Q_v^{(k)}$  and  $Q_i^{(k)}$ . More specifically, the complete trees of distance  $k$  from  $t$  are in one-to-one correspondence with the pairs of nonzero elements which occur in corresponding positions in  $Q_v^{(k)}$  and  $Q_i^{(k)}$ . Since corresponding pairs of nonzero entries are used to find the complete trees, there are no complete trees found from the  $k$ th compound when either  $Q_v^{(k)} = 0$  or  $Q_i^{(k)} = 0$  or  $Q_v^{(k)} = Q_i^{(k)} = 0$ . Therefore, the highest compound required for finding all of the complete trees

is  $l(p_1, p_2)$ , where  $p_1$  is the rank of the cotree of  $t$  relative to  $G_v$ , and  $p_2$  is the rank of the cotree of  $t$  relative to  $G_i$ .

Now that it has been shown that the complete trees can be found by the compound method, interest is concentrated on finding their signs. Consider a pair of nonzero elements corresponding to a complete tree  $t_j$  of distance  $k$  from  $t$ . From (3-4), the element in  $Q_v^{(k)}$  is  $(\det A_{vt})(\det A_{vt_k})$ , where  $A_{vt_k}$  is formed by replacing a set of  $k$  columns of  $A_{vt}$  by a set of  $k$  columns of  $A_{vc}$ . The corresponding element in  $Q_i^{(k)}$  is  $(\det A_{it})(\det A_{it_k})$ , where  $A_{it_k}$  is formed by replacing the same  $k$  columns of  $A_{it}$  by those columns of  $A_{ic}$  which correspond to the columns of  $A_{vc}$  appearing in  $A_{vt_k}$ . Thus, the product,  $\pi_j$ , of the corresponding terms from  $Q_v^{(k)}$  and  $Q_i^{(k)}$  may be written in the form

$$\pi_j = (\det A_{vt})(\det A_{it})(\det A_{vt_k})(\det A_{it_k}) \quad (5-4)$$

Matrices  $A_{vt_k}$  and  $A_{it_k}$  differ from matrices  $A_{vt_j}$  and  $A_{it_j}$  of (5-1), respectively, by at most a column rearrangement. From the column ordering in (5-2), the same column rearrangement applies to both matrices. Thus,

$$\pi_j = (\det A_{vt})(\det A_{it}) \epsilon_j \quad (5-5)$$

The product of the first two terms of (5-5) is, by definition, the sign  $\epsilon_t$  of the original known complete tree  $t$ . Thus

$$\pi_j = \epsilon_t \epsilon_j \quad (5-6)$$

If  $t$  contains only ordinary edges, then  $\epsilon_t = +1$ , and the product of the corresponding nonzero entries of  $Q_V^{(k)}$  and  $Q_I^{(k)}$  gives the correct sign for all complete trees of distance  $k$  from  $t$ . If  $t$  contains active edges,  $\epsilon_t$  is either  $+1$  or  $-1$ . When  $\epsilon_t = +1$ , the products of corresponding elements in all compounds give the correct signs. When  $\epsilon_t = -1$ , all signs are incorrect, but easily changed. Thus all complete trees of a graph pair and their signs can be found from the compounds of  $Q_V$  and  $Q_I$ . The procedure is demonstrated by the following example.

Example 5-1. Figures 5-1(b) and (c) show  $G_V$  and  $G_I$  for the network of Figure 5-1(a). The starting tree  $t$  is cfab which contains only ordinary edges. Omitting all columns of the compounds which are known to be null or of no use (by inspection of the cotrees of  $t$  relative to  $G_V$  and  $G_I$ ) gives

$$Q_V^{(1)} = \begin{matrix} & \begin{matrix} d & e & h & m & g \end{matrix} \\ \begin{matrix} c \\ f \\ a \\ b \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \textcircled{1} & 0 & 1 & 0 & \textcircled{1} \\ \textcircled{1} & 0 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$Q_I^{(1)} = \begin{matrix} & \begin{matrix} d & e & h & m & g \end{matrix} \\ \begin{matrix} c \\ f \\ a \\ b \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \textcircled{1} & 0 & 0 & 1 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & \textcircled{1} & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$Q_v^{(2)} = \begin{matrix} & \begin{matrix} de & dg & eg & hm & hg & mg \end{matrix} \\ \begin{matrix} cf \\ ca \\ cb \\ fa \\ fb \\ ab \end{matrix} & \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ \textcircled{1} & 0 & -1 & \textcircled{1} & 0 & -1 \\ \textcircled{1} & 0 & 0 & \textcircled{1} & 0 & 0 \end{bmatrix} \end{matrix} *$$

$$Q_i^{(2)} = \begin{matrix} & \begin{matrix} de & dg & eg & hm & hg & mg \end{matrix} \\ \begin{matrix} cf \\ ca \\ cb \\ fa \\ fb \\ ab \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ \textcircled{1} & 0 & -1 & -1 & -1 & 0 \\ \textcircled{1} & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \end{matrix} *$$

\* (columns dh, dm, em, eh are deleted, for dh and em of the voltage graph and dm and eh of the current graph constitute parallel edges.)

$$Q_v^{(3)} = \begin{array}{cc} & \begin{array}{cc} \text{deg} & \text{hmg} \end{array} \\ \begin{array}{c} \text{cfa} \\ \text{cfb} \\ \text{cab} \\ \text{fab} \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ \textcircled{1} & \textcircled{1} \end{bmatrix} \end{array} **$$

$$Q_i^{(3)} = \begin{array}{cc} & \begin{array}{cc} \text{deg} & \text{hmg} \end{array} \\ \begin{array}{c} \text{cfa} \\ \text{cfb} \\ \text{cab} \\ \text{fab} \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \textcircled{1} & \textcircled{1} \end{bmatrix} \end{array} **$$

\*\* (For the same reason as \*, columns deh, dem, dhm, dhg, dm, g, ehm, ehg are deleted.)

where the corresponding pairs of nonzero entries in each compound are circled. The complete trees and their signs are then: cfab+ (the starting tree); cdab+, cgab+, cfdb+, cfae+ (trees of distance one from cfab); cdgb+, cdae+, ceag+, cham-, cfde+, cfhm-, (trees of distance two from cfab); and cdeg+, chmg- (trees of distance three from cfab).

Many practical active networks give rise to graphs having many ordinary edges. In such cases a certain inefficiency is apparent in this tree-finding method. Duplicate columns in the compound-pairs are generated for complete trees containing only ordinary edges, even though no column comparison is required. This is clearly seen in the example above. In  $Q_v^{(1)}$  and  $Q_i^{(1)}$ ,

the columns labeled d (also e) are identical. In  $Q_v^{(2)}$  and  $Q_i^{(2)}$ , columns de are identical. A modification of the method which eliminates this difficulty involves defining a matrix  $Q$  for the so-called complete graph. In the complete graph, a single edge represents each passive element and edge pair represents each active element. Thus a single graph rather than a graph pair is used. Figure 5-1(d) shows the complete graph  $G$  for the network of the preceding example. The starting point is a complete tree,  $t$ , of  $G$  containing only ordinary edges. Matrix  $Q$ , as usual, is obtained from the f-cut-set matrix of  $G$  with respect to  $t$ . Complete trees containing active edges and their signs are found by searching for corresponding nonzero entries in column pairs of each compound. The product of the elements in each pair gives the correct sign. Trees containing only passive edges are found by inspecting certain columns in each compound.

Since trees containing both voltage and current edges are inadmissible, only certain columns of each compound need be calculated. Also, it is useful to rearrange the calculated columns of the compounds so that columns corresponding to related voltage and current edges are adjacent. The rearranged  $k$ th compound with columns deleted as explained above is denoted by  $\bar{Q}^{(k)}$ . The problem of Example 5-1 is now solved by this modification using the same starting tree. The matrices are

$$\bar{Q}(1) = \begin{matrix} & d & e & h & h' & m & m' & g & g' \\ \begin{matrix} c \\ f \\ a \\ b \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \textcircled{1} & 0 & 1 & 0 & 0 & 1 & \textcircled{1} & \textcircled{1} \\ \textcircled{1} & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\bar{Q}(2) = \begin{matrix} & de & dg & dg' & eg & eg' & hm & hm' & hg & hg' & mg & mg' \\ \begin{matrix} cf \\ ca \\ cb \\ fa \\ fb \\ ab \end{matrix} & \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & \textcircled{-1} & \textcircled{-1} \\ \textcircled{1} & 0 & 0 & \textcircled{-1} & \textcircled{-1} & \textcircled{1} & \textcircled{-1} & 0 & -1 & -1 & 0 \\ \textcircled{1} & 0 & 0 & 0 & 0 & \textcircled{1} & \textcircled{-1} & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} *$$

$$\bar{Q}(3) = \begin{matrix} & deg & deg' & hmg & h'm'g' \\ \begin{matrix} cfa \\ cfb \\ cab \\ fab \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \textcircled{1} & \textcircled{1} & \textcircled{1} & \textcircled{1} \end{bmatrix} \end{matrix} **$$

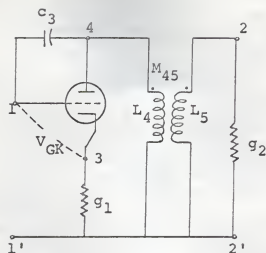
\* (Since  $dh$ ,  $em$ ,  $d'm'$  and  $e'h'$  consist of parallel edges, columns  $dh$ ,  $d'h'$ ,  $em$ ,  $e'm'$ ,  $dm$ ,  $d'm'$ ,  $eh$ , and  $e'h'$  are deleted.)

\*\* (For the same reason as \*, the columns  $deh$ ,  $d'e'h'$ ,  $dem$ ,  $d'e'm'$ ,  $d'hm$ ,  $d'h'm'$ ,  $d'hg$ ,  $d'h'g'$ ,  $dmg$ ,  $d'm'g'$ ,  $ehm$ ,  $e'h'm'$ ,  $ehg$  and  $e'h'g'$  are deleted.)

where the primed column labels denote the current edges. The circled nonzero entries correspond to the complete trees, with



circled pairs indicating complete trees containing active edges.  
The product of each pair gives the sign associated with the  
complete tree.



(a) Active Network

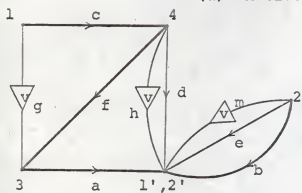
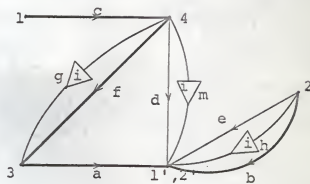
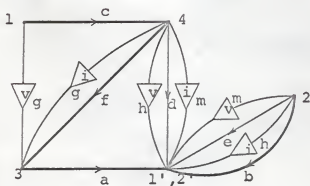
(b) Voltage Graph,  $G_V$ (c) Current Graph,  $G_i$ (d) Complete Graph,  $G$ .

Fig. 5-1.

## VI. CONCLUSIONS

A new method for generating trees and signed tree pairs was described which involves searching for nonzero entries in the various compounds of a matrix  $Q$ . This compound method was shown to be inherently inefficient as a tree-generating procedure. In attempting to modify the method so as to improve its efficiency some very interesting new results were discovered.

It was found that the familiar idea of pivotal condensation can be used to generate the  $k$ th compound of a matrix in terms of minors of order two from the  $k-1$  compound and nonzero elements from the  $k-2$  compound. Thus, all compounds of a given matrix can be obtained by calculating only minors of order two. This result applies to compounds of any matrix of elements defined over a field, and is particularly simple for the classes of matrices used in network theory. Since the use of compound matrices in network theory is quite new, there is a possibility that this result may be applicable to a variety of new problems.

Deo has recently pointed out that the number of calculations can be reduced if a particular type of tree, called a central tree, is used as a starting tree in tree-generating procedures. The concept of the central tree was derived here using compound matrix theory. This new derivation leads to further reductions in calculations made possible by using the central tree, which were not apparent from Deo's approach. These simplifications result in improvements in relatively efficient tree-generating procedures given by Hakimi and Mayeda.

A new theorem was stated and proved which makes it possible to easily find a starting tree in certain special cases such that the labor of finding the other trees is significantly reduced. Further exploration of this theorem leads to an extension of the class of cases to which it applies by using a duality argument. This theorem also leads to a method for finding a central tree of a complete graph. The results again, can be used to improve existing tree-generating procedures; and they represent some initial progress in the general unsolved problem of finding a central tree for any graph.

The compound method, inefficient in itself as a tree-generating procedure, presents a new perspective in which to view the general tree finding problem. The compact notation and high degree of organization of the compound method lead to improvements in existing, more efficient, tree-generating procedures which were not previously known. The method also makes it possible to easily visualize and find certain subsets of the set of all trees; a property which should become increasingly useful as a better understanding of the structure of active networks is developed. Finally, many of the concepts arising from studying the compound method are given a useful geometrical interpretation in terms of Cummins' tree-graph. In turn, certain properties of the tree graph can be derived by means of the compound approach.

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USE OF COMPOUND MATRICES IN TOPOLOGICAL ANALYSIS

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A new method, the compound matrix method, is presented for systematically finding the trees of a graph or the complete trees of a graph pair without duplication. The signs associated with active tree pairs are generated during the tree-finding procedure with negligible additional labor.

It is well-known that the trees of a graph  $G$  of distance one from a starting tree  $t$  are in one-to-one correspondence with the nonzero entries of  $Q$ , where  $Q_f = [U \ Q]$  is the  $f$ -cut-set matrix of  $G$  relative to  $t$ . Here it is proved that the trees of distance  $k$  from  $t$  are in one-to-one correspondence with the nonzero entries in  $Q^{(k)}$ , the  $k$ th compound of  $Q$ . Thus, all trees may be found by inspection of  $Q$  and its nonvanishing compounds.

The complete trees associated with a graph pair may be found by starting with a given complete tree and searching for corresponding nonzero entries in compound pairs  $Q_v^{(k)}$  and  $Q_i^{(k)}$ , which are derived from the voltage and current graph, respectively. If the starting tree pair contains active edges, the product of corresponding nonzero entries must be multiplied by the sign of the starting tree pair. When an active network contains a large number of passive elements, the complete trees and their signs are more efficiently found by using a single graph containing designated active edge pairs. A matrix  $Q$  is first determined using a starting tree containing only passive edges. Active tree pairs are located by inspecting certain column pairs in  $Q^{(k)}$  for corresponding nonzero entries, with the correct sign being given by the product of the entries. Trees containing only passive edges

are found by inspecting certain columns in each compound.

A number of methods for reducing the labor in calculations are explored. Some of these result in improvements in more classical tree-generating procedures.